# Lindley Approximation Technique for the Parameters of Lomax Distribution

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**Abstract:** The present study is concerned with the estimation of shape and scale parameter of Lomax distribution using Bayesian approximation techniques (Lindley's Approximation). Different priors viz gamma, exponential and Levy priors are used to obtain the Bayes estimates of parameters of Lomax distributions under Lindley approximation technique. For comparing the efficiency of the obtained results a simulation study is carried out using R-software.

**Keywords:** Lomax distribution, Bayesian Estimation, Prior, Loss functions, Lindley's Approximation.

## 1. INTRODUCTION

The Lomax distribution also known as Pareto distribution of second kind has, in recent years, assumed opposition of importance in the field of life testing because of its uses to fit business failure data. It has been used in the analysis of income data, and business failure data. It may describe the lifetime of a decreasing failure rate component as a heavy tailed alternative to the exponential distribution. Lomax distribution was introduced by Lomax (1954), Abdullah and Abdullah (2010) [1, 2] estimated the parameters of Lomax distribution based on generalized probability weighted moment. Zangan (1999) [3] deals with the properties of the Lomax distribution with three parameters. Abd-Elfatth and Mandouh (2004) [4] discussed inference for R = Pr{Y<X} when X and Y are two independent Lomax random variables. Afaq *et al*. (2015) [5] performs comparisons of maximum likelihood estimation (MLE) and Bayes estimates of shape parameter using prior distribution. Afaq *et al*. [6] proposed Length biased Weighted Lomax distribution and discussed its structural properties. The cumulative distribution function of Lomax distribution is given by

$$F\left(x;\alpha,\beta\right)=1-(1+\beta x)^{-\alpha}, \quad x>0,\ \ \alpha,\beta>0, \tag{1.1}$$

and the corresponding probability density function is given by

$$f\left(x;\alpha,\beta\right)=\alpha\beta(1+\beta x)^{-(\alpha+1)},\ \ x>0,\ \ \alpha,\beta>0, \tag{1.2}$$

where $\alpha$ and $\beta$ are the shape and scale parameters respectively.

The Lomax distribution has not been discussed in detail under the Bayesian approach. The Bayesian paradigm is conceptually simple and probabilistically elegant. Sometimes posterior distribution is expressible in terms of complicated analytical function and requires intensive calculation because of its numerical implementations. It is therefore useful to study approximate and large sample behavior of posterior distribution. Uzma (2017) [7] obtains the estimate of shape parameter of inverse Lomax distribution. Our present study aims to obtain the estimate the shape and scale parameters of Lomax distribution using Lindley approximation technique using Gamma prior, Exponential prior and Inverse Levy prior. A simulation study has also been conducted along with concluding remarks.

## 2. LINDLEY APPROXIMATION

Sometimes, the integrals appearing in Bayesian estimation can't be reduced to closed form and it becomes tedious to evaluate of the posterior expectation for obtaining the Baye's estimators. Thus, we propose the use of Lindley's approximation method (1980) [8] for obtaining Baye's estimates. Lindley developed an asymptotic approximation to the ratio

$$I(X)=\frac{\int_{(\alpha,\beta)}U(\alpha,\beta)e^{L(\alpha,\beta)+\rho(\alpha,\beta)}\partial(\alpha,\beta)}{\int_{(\alpha,\beta)}e^{L(\alpha,\beta)+\rho(\alpha,\beta)}\partial(\alpha,\beta)},$$

where $U(\alpha,\beta)$ is function of $\alpha$ and $\beta$ only and $L(\alpha,\beta)$ is the log-likelihood and $\rho(\alpha,\beta)=\log g\left(\alpha,\beta\right)$. Let $(\hat{\alpha},\hat{\beta})$ denotes the MLE of $(\alpha,\beta)$. For sufficiently large sample size n, using the approach developed by Lindley (1980) [8], the ratio of integral $I(X)$ as defined above can be written as

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$$I(X) = u(\alpha,\beta) + \frac{1}{2}(u_{11}\phi_{11} + u_{22}\phi_{22}) + \rho_1 u_1 \phi_{11} + \rho_2 u_2 \phi_{22} + \frac{1}{2}\left(L_{30}u_1\phi^2{}_{11} + L_{03}u_2\phi^2{}_{22} + (L_{21}u_1 + L_{12}u_2)\phi_{11}\phi_{22}\right) \quad (2.1)$$

Now $L_{20} = \frac{\partial^2 l}{\partial\alpha^2} = \frac{-n}{\alpha^2}$, $L_{30} = \frac{\partial^3 l}{\partial\alpha^3} = \frac{2n}{\alpha^3}$, $L_{21} = \frac{\partial^3 l}{\partial\beta\,\partial\alpha^2} = 0$,

$$L_{02} = \frac{\partial^2 l}{\partial\beta^2} = -\frac{n}{\beta^2} + (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2,$$

$$L_{03} = \frac{\partial^3 l}{\partial\beta^3} = \frac{2n}{\beta^3} + 2(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3, \text{ and}$$

$$L_{12} = \frac{\partial^3 l}{\partial\alpha\,\partial\beta^2} = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$$

Hence, $\phi_{11} = -L_{20}{}^{-1} = \frac{\alpha^2}{n}$

and $\phi_{22} = -L_{02}{}^{-1} = \left[\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right]^{-1}$.

Also, $\rho = \log g_{12}(\alpha,\beta) = (a_1-1)\log\alpha - a_2\alpha + (b_1-1)\log\beta - b_2\beta$

Several authors have used this approximation for obtaining the Bayes estimators for some lifetime distributions; see among others, Sing *et al* (2008) [9]. We have used three different loss function for estimating the parameters, LINEX loss function given by $\hat{\alpha}_{LLF} = -\frac{1}{c}\log E(e^{-c\alpha})$ where $E_\alpha$ is the Posterior expectation.

Generalized Entropy loss function is given by $\hat{\alpha}_{GELF} = \left[E_\alpha(\alpha^{-c})\right]^{-1/c}$ provided $E_\alpha(\alpha^{-c})$ exists and is finite.

Squared error loss function (SELF) is given as $\hat{\alpha}_{SELF} = u = u(\alpha,\beta)$

## 3. LINDLEY APPROXIMATION UNDER THE ASSUMPTION OF GAMMA PRIOR

In this section we consider the Bayes estimation of the unknown parameter(s). When both are unknown, it is assumed that $\alpha\, and\, \beta$ have the following gamma prior distributions;

$$g_1(\alpha) \propto \alpha^{a_1-1}e^{-a_2\alpha}, \alpha > 0, a_1, a_2 > 0, \quad (3.1)$$

$$g_2(\beta) \propto \beta^{b_1-1}e^{-b_2\beta}, \beta > 0, b_1, b_2 > 0. \quad (3.2)$$

Here all the hyper parameters $a_1, a_2, b_1, b_2$ are assumed to be known and non-negative.

Then the joint prior density is defined by

$$g_{12}(\alpha,\beta) \propto \alpha^{a_1-1}\beta^{b_1-1}e^{-a_2\alpha}e^{-b_2\beta}, \alpha,\beta > 0, b_1, b_2 > 0. \quad (3.3)$$

According to Bayes theorem, the joint posterior distribution of the parameters of $\alpha\, and\, \beta$ is

$$\pi(\alpha,\beta \mid x) \propto L(x \mid \alpha,\beta) \times g_{12}(\alpha,\beta)$$

$$\pi(\alpha,\beta \mid x) = K\alpha^{n+a_1-1}\beta^{n+b_1-1}\prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)}e^{-a_2\alpha}e^{-b_2\beta}$$

where *K* is a normalizing constant defined as

$$K^{-1} = \int_0^\infty\int_0^\infty \alpha^{n+a_1-1}\beta^{n+b_1-1}\prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)}e^{-a_2\alpha}e^{-b_2\beta}\,d\alpha\, d\beta. \quad (3.4)$$

### 3.1. Estimate of α and β under Different Loss Functions

The Bayes estimators are derived under the assumption of gamma prior using the following three different loss functions:

#### *Under LINEX Loss Function*

$$E(\alpha \mid x) = e^{-c\hat{\alpha}}\left[1 + \frac{c\hat{\alpha}}{n}\left(\frac{c\hat{\alpha}}{2} - a_1 + a_2\hat{\alpha} - \frac{\hat{\alpha}}{2}t_1 t_2\right)\right] \quad (3.5)$$

where, $t_1 = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$ and $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$

$$E(\beta \mid x) = e^{-c\hat{\beta}}\left[1 + ct_2\left(\frac{c}{2} - \frac{(b_1 - 1 - b_2\beta)}{\hat{\beta}} - t_3 t_2\right)\right] \quad (3.6)$$

where, $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

#### *Under Generalized Entropy Loss Function*

$$E(\alpha \mid x) = \hat{\alpha}^{-c}\left[1 + \frac{c}{n}\left(\frac{(c+1)}{2} - a_1 + a_2\hat{\alpha} - \frac{\hat{\alpha}}{2}t_1 t_2\right)\right] \quad (3.7)$$

where, $t_1 = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$ and $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$

$$E(\beta \mid x) = \hat{\beta}^{-c}\left[1 + \frac{c}{\hat{\beta}}t_2\left(\frac{(c+1)}{2\hat{\beta}} - \frac{(b_1 - 1 - b_2\beta)}{\hat{\beta}} - t_3 t_2\right)\right] \tag{3.8}$$

where $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

***Under Squared Error Entropy Loss Function***

$$E(\alpha \mid x) = \hat{\alpha}\left[1 + \frac{1}{n}\left(a_1 - a_2\hat{\alpha} - \frac{\hat{\alpha}}{2}t_1 t_2\right)\right] \tag{3.9}$$

where, $t_1 = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$ and $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$

$$E(\beta \mid x) = \hat{\beta} + t_2\left[\left(\frac{(b_1 - 1 - b_2\beta)}{\hat{\beta}} + t_3 t_2\right)\right] \tag{3.10}$$

where $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

## 4. LINDLEY APPROXIMATION UNDER THE ASSUMPTION OF EXPONENTIAL PRIOR

Assuming that $\alpha$ and $\beta$ has a Exponential prior with known hyper-parameter $d_1$ and $d_2$ respectively given as

$$g_3(\alpha) \propto e^{-d_1\alpha} \quad ,\alpha > 0, \tag{4.1}$$

$$g_4(\beta) \propto e^{-d_2\beta} \quad ,\beta > 0, \tag{4.2}$$

Then the joint prior density is defined by

$$g_{34}(\alpha,\beta) \propto e^{-d_1\alpha - d_2\beta}. \tag{4.3}$$

The joint posterior distribution of $\alpha, \beta$ is given by

$$\pi_2(\alpha,\beta \mid x) \propto L(x \mid \alpha,\beta) \times g_{34}(\alpha,\beta)$$

$$\pi_2(\alpha,\beta \mid x) = K\alpha^n\beta^n \prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)} e^{-d_1\alpha} e^{-d_2\beta}$$

where $K$ is a normalizing constant defined as

$$K^{-1} = \int_0^\infty\int_0^\infty \alpha^n\beta^n \prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)} e^{-d_1\alpha} e^{-d_2\beta} d\alpha d\beta. \tag{4.4}$$

Again, $g_{34}(\alpha,\beta) \propto e^{-d_1\alpha - d_2\beta}$

$$\therefore \qquad \rho = \ln g_{34}(\alpha,\beta) = -d_1\alpha - d_2\beta$$

$$\Rightarrow \qquad \rho_1 = -d_1 \text{ and } \rho_2 = -d_2$$

### 4.1. Estimate of α and β under Different Loss Functions

The Bayes estimators are derived under the assumption of exponential prior using the following three different loss functions:

***Under LINEX Loss Function***

$$E(\alpha \mid x) = e^{-c\hat{\alpha}}\left[1 + \frac{c\hat{\alpha}}{n}\left(\frac{c\hat{\alpha}}{2} + d_1\hat{\alpha} - 1 - \frac{\hat{\alpha}}{2}t_1 t_2\right)\right] \tag{4.5}$$

where, $t_1 = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$ and

$$t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$$

$$E(\beta \mid x) = e^{-c\hat{\beta}}\left[1 + ct_2\left(\frac{c}{2} + d_2 - t_3 t_2\right)\right] \tag{4.6}$$

where $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

***Under Generalized Entropy Loss Function***

$$E(\alpha \mid x) = \hat{\alpha}^{-c}\left[1 + \frac{c}{n}\left(\frac{(c+1)}{2} + d_1 - 1 - \frac{\hat{\alpha}}{2}t_1 t_2\right)\right] \tag{4.7}$$

where, $t_1 = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$ and $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$

$$E(\beta \mid x) = \hat{\beta}^{-c}\left[1+\frac{c}{\hat{\beta}}t_2\left(\frac{(c+1)}{2\hat{\beta}}+d_2-t_3t_2\right)\right] \tag{4.8}$$

where $t_2 = \dfrac{1}{\left(\dfrac{n}{\beta^2}-(\alpha+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3}-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

***Under Squared Error Loss Function***

$$E(\alpha \mid x) = \hat{\alpha}\left[1+\frac{1}{n}\left(1-d_1+\frac{\hat{\alpha}}{2}t_1t_2\right)\right] \tag{4.9}$$

where, $t_1 = \sum_{i=1}^{n}\left(\dfrac{x_i}{1+\beta x_i}\right)^2$ and

$$t_2 = \frac{1}{\left(\dfrac{n}{\beta^2}-(\alpha+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{1+\beta x_i}\right)^2\right)}$$

$$E(\beta \mid x) = \hat{\beta}+t_2\left[\left(-d_2+t_3t_2\right)\right] \tag{4.10}$$

where $t_2 = \dfrac{1}{\left(\dfrac{n}{\beta^2}-(\alpha+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3}-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

## 5. LINDLEY APPROXIMATION UNDER THE ASSUMPTION OF INVERSE LEVY PRIOR

Assuming that $\alpha$ and $\beta$ has a Inverse Levy prior with known hyper-parameter $r_1$ and $r_2$ given by

$$g_5(\alpha) \propto \alpha^{\frac{-1}{2}}e^{-r_1\frac{\alpha}{2}} \qquad ,\alpha>0, \tag{5.1}$$

$$g_6(\beta) \propto \beta^{\frac{-1}{2}}e^{-r_2\frac{\beta}{2}} \qquad ,\beta>0, \tag{5.2}$$

Then the joint prior density is defined by

$$g_{56}(\alpha,\beta) \propto \alpha^{\frac{-1}{2}}\beta^{\frac{-1}{2}}e^{-r_1\frac{\alpha}{2}-r_2\frac{\beta}{2}} \tag{5.3}$$

The joint posterior distribution of $\alpha,\beta$ is given by

$$\pi_2(\alpha,\beta \mid x) \propto L(x \mid \alpha,\beta)\times g_{56}(\alpha,\beta)$$

$$\pi_2(\alpha,\beta \mid x) = K\alpha^{n-\frac{1}{2}}\beta^{n-\frac{1}{2}}\prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)}e^{-r_1\frac{\alpha}{2}-r_2\frac{\beta}{2}}$$

where $K$ is a normalizing constant defined as

$$K^{-1} = \int_0^\infty\int_0^\infty \alpha^{n-\frac{1}{2}}\beta^{n-\frac{1}{2}}\prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)}e^{-r_1\frac{\alpha}{2}-r_2\frac{\beta}{2}}d\alpha d\beta. \tag{5.4}$$

Again, $g_{56}(\alpha,\beta) \propto \alpha^{-\frac{1}{2}}\beta^{-\frac{1}{2}}e^{-r_1\frac{\alpha}{2}-r_2\frac{\beta}{2}}$

$$\therefore \qquad \rho = \ln g_{56}(\alpha,\beta) = -\frac{1}{2}\log\alpha-\frac{1}{2}\log\beta-\frac{\alpha}{2}r_1-\frac{\beta}{2}r_2$$

$$\Rightarrow \qquad \rho_1 = -\frac{1}{2}\left(\frac{1}{\alpha}+r_1\right) and \ \rho_2 = -\frac{1}{2}\left(\frac{1}{\beta}+r_2\right)$$

### 4.1. Estimate of α and β under Different Loss Functions

The Bayes estimators are derived under the assumption of inverse levy prior using the following three different loss functions:

**Under LINEX Loss Function**

$$E(\alpha \mid x) = e^{-c\hat{\alpha}}\left[1+\frac{c\hat{\alpha}}{n}\left(\frac{c\hat{\alpha}}{2}+\frac{1}{2}(1+r_1\hat{\alpha})-1-\frac{\hat{\alpha}}{2}t_1t_2\right)\right] \tag{5.5}$$

Where, $t_1 = \sum_{i=1}^{n}\left(\dfrac{x_i}{1+\beta x_i}\right)^2$ and

$$t_2 = \frac{1}{\left(\dfrac{n}{\beta^2}-(\alpha+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{1+\beta x_i}\right)^2\right)}$$

$$E(\beta \mid x) = e^{-c\hat{\beta}}\left[1+ct_2\left(\frac{c}{2}+\frac{(1+r_2\hat{\beta})}{\hat{\beta}}-t_3t_2\right)\right] \tag{5.6}$$

where $t_2 = \dfrac{1}{\left(\dfrac{n}{\beta^2}-(\alpha+1)\sum_{i=1}^{n}\left(\dfrac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3}-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

***Under Generalized Entropy Loss Function***

$$E(\alpha \mid x) = \hat{\alpha}^{-c}\left[1+\frac{c}{n}\left(\frac{(c+1)}{2}+\frac{1}{2}(1+r_1\alpha)-1-\frac{\hat{\alpha}}{2}t_1t_2\right)\right] \tag{5.7}$$

where, $t_1 = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$ and $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$

$$E(\beta \mid x) = \hat{\beta}^{-c}\left[1 + \frac{c}{\hat{\beta}} t_2 \left(\frac{(c+1)}{2\hat{\beta}} + \frac{(1+r_2\hat{\beta})}{2\hat{\beta}} - t_3 t_2\right)\right] \tag{5.8}$$

where $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

### *Under Squared Error Loss Function*

$$E(\alpha \mid x) = \hat{\alpha}\left[1 + \frac{1}{n}\left(-\frac{(1+\alpha r_1)}{2} + 1 + \frac{\hat{\alpha}}{2} t_1 t_2\right)\right] \tag{5.9}$$

where, $t_1 = \sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2$ and

$$t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$$

$$E(\beta \mid x) = \hat{\beta} + t_2\left[\frac{-(1+r_2\hat{\beta})}{2\hat{\beta}} + t_3 t_2\right] \tag{5.10}$$

where $t_2 = \frac{1}{\left(\frac{n}{\beta^2} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2\right)}$ and

$$t_3 = \left(\frac{n}{\beta^3} - (\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^3\right)$$

## 5. SIMULATION STUDY

In our simulation study, we chose a sample size of n=25, 50 and 100 to represent small, medium and large data set. The scale parameter and shape parameter is estimated for Lomax distribution by using Bayesian method of estimation under Gamma prior, Exponential prior and Inverse Levy prior. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes by using various types of loss functions. We have chosen the initial values of parameters were $\alpha = 0.5, 1.0 \,\&\, 1.5$ and $\beta = 1.0, 1.5 \,\&\, 2.0$. The values of hyper parameters are $a_1 = b_1 = 0.3, 0.6, \&\, 0.9$; $a_2 = b_2 = 0.2, 0.5, \&\, 0.8$; $d_1 = d_2 = 0.2, 0.6, \&\, 1.0$ and $r_1 = r_2 = 0.1, 0.5 \,\&\, 0.9$. The value of loss parameter $c = \pm 0.5$. The results are presented in tables for different selections of the parameters.

**Table 1: Bayes Estimates and MSE (in Parenthesis) for α and β using Gamma Prior** $(a_1 = b_1 = 0.3, 0.6, \&\, 0.9)$, $(a_2 = b_2 = 0.2, 0.5, \&\, 0.8)$

| n | $\hat{\alpha}_{SELF}$ | $\hat{\beta}_{SELF}$ | $\hat{\alpha}_{LLF}$ | | $\hat{\beta}_{LLF}$ | | $\hat{\alpha}_{GELF}$ | | $\hat{\beta}_{GELF}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 25 | 0.5004 (0.0233) | 0.9606 (0.0047) | 0.7795 (0.1014) | 1.2859 (0.6410) | 0.6182 (0.1488) | 1.6158 (0.3824) | 1.4347 (0.8970) | 0.7039 (0.0649) | 1.0188 (0.0035) | 0.9805 (0.0035) |
| | 0.9935 (0.2362) | 1.4575 (0.0405) | 0.6115 (0.3871) | 1.6516 (0.6608) | 0.4821 (1.0747) | 2.0710 (0.3649) | 1.0182 (0.2365) | 0.9917 (0.2362) | 0.8275 (0.4909) | 1.2076 (0.1242) |
| | 1.4632 (0.2459) | 1.9514 (0.2965) | 0.4863 (1.2720) | 2.1019 (0.6068) | 0.3765 (2.9299) | 2.6500 (0.7168) | 0.8387 (0.6818) | 1.2036 (0.3324) | 0.7152 (1.9447) | 1.3973 (0.6574) |
| 50 | 0.5002 (0.0074) | 0.9763 (0.9416) | 0.7791 (0.0853) | 1.2849 (0.6236) | 0.6136 (1.0903) | 1.6290 (1.3368) | 1.4244 (0.8620) | 0.7055 (0.0496) | 1.0115 (0.9411) | 0.9882 (0.9411) |
| | 0.9968 (0.0779) | 1.4754 (0.1461) | 0.6089 (0.2308) | 1.6502 (0.5008) | 0.4781 (1.1897) | 2.0906 (0.4943) | 1.0090 (0.0780) | 0.9959 (0.0779) | 0.8230 (0.6037) | 1.2147 (0.2268) |
| | 1.4818 (0.0935) | 0.9606 (0.0047) | 0.4793 (1.1350) | 2.1096 (0.4649) | 0.3725 (2.8355) | 2.6828 (0.6531) | 0.8275 (0.5453) | 1.2142 (0.1748) | 0.7115 (1.8470) | 1.4052 (0.5406) |
| 100 | 0.5001 (0.0031) | 0.9896 (0.4392) | 0.7789 (0.0810) | 1.2845 (0.6186) | 0.6096 (0.5914) | 1.6401 (0.8488) | 1.4193 (0.8482) | 0.7063 (0.0457) | 1.0051 (0.4391) | 0.9948 (0.4391) |
| | 0.9984 (0.0387) | 1.4878 (0.0622) | 0.6077 (0.1926) | 1.6495 (0.4606) | 0.4752 (1.1122) | 2.1040 (0.4270) | 1.0045 (0.0388) | 0.9979 (0.0387) | 0.8197 (0.5248) | 1.2198 (0.1405) |
| | 1.4909 (0.2942) | 1.9877 (0.0229) | 0.4758 (1.3431) | 2.1133 (0.6704) | 0.3701 (2.6792) | 2.7014 (0.5148) | 0.8220 (0.7538) | 1.2195 (0.3728) | 0.7092 (1.6888) | 1.4098 (0.3710) |

**Table 2: Bayes Estimates and MSE (in Parenthesis) for $\alpha$ and $\beta$ using Exponential Prior** $(d_1 = d_2 = 0.2, 0.6, \& 1.0)$

| n | $\hat{\alpha}_{SELF}$ | $\hat{\beta}_{SELF}$ | $\hat{\alpha}_{LLF}$ | | $\hat{\beta}_{LLF}$ | | $\hat{\alpha}_{GELF}$ | | $\hat{\beta}_{GELF}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 25 | 0.5124 (0.0234) | 0.9591 (0.2144) | 0.7741 (0.0984) | 1.2949 (0.6552) | 0.6187 (0.3581) | 1.6145 (0.5904) | 1.4177 (0.8656) | 0.7123 (0.0684) | 1.0196 (0.2131) | 0.9798 (0.2132) |
| | 1.0055 (0.2362) | 1.4569 (0.0605) | 0.6078 (0.3899) | 1.6615 (0.6738) | 0.4823 (1.0943) | 2.0704 (0.3840) | 1.0122 (0.2363) | 0.9977 (0.2362) | 0.8277 (0.5106) | 1.2074 (0.1442) |
| | 1.4812 (0.2449) | 1.9524 (0.1959) | 0.4891 (1.2663) | 2.0892 (0.5917) | 0.3763 (2.8299) | 2.6514 (0.6179) | 0.8338 (0.6883) | 1.2109 (0.3281) | 0.7150 (1.8446) | 1.3976 (0.5564) |
| 50 | 0.5062 (0.0074) | 0.9758 (0.9416) | 0.7764 (0.0838) | 1.2894 (0.6307) | 0.6138 (1.0902) | 1.6286 (1.3362) | 1.4159 (0.8464) | 0.7097 (0.0514) | 1.0117 (0.9411) | 0.9880 (0.9411) |
| | 1.0028 (0.0779) | 1.4752 (0.1461) | 0.6071 (0.2322) | 1.6552 (0.5072) | 0.4781 (1.1896) | 2.0904 (0.4941) | 1.0060 (0.0779) | 0.9989 (0.0779) | 0.8230 (0.6037) | 1.2147 (0.2269) |
| | 1.4908 (0.0933) | 1.9746 (0.1875) | 0.4807 (1.1321) | 2.1032 (0.4571) | 0.3724 (2.8357) | 2.6831 (0.6536) | 0.8251 (0.5486) | 1.2179 (0.1727) | 0.7114 (1.8471) | 1.4053 (0.5405) |
| 100 | 0.5031 (0.0031) | 0.9895 (0.4392) | 0.7776 (0.0802) | 1.2867 (0.6221) | 0.6097 (0.5914) | 1.6400 (0.8487) | 0.8207 (0.1060) | 1.2213 (0.5235) | 1.0051 (0.4391) | 0.9947 (0.4391) |
| | 1.0014 (0.0387) | 1.4878 (0.0622) | 0.6068 (0.1933) | 1.6519 (0.4638) | 0.4752 (1.1122) | 2.1040 (0.4269) | 1.0030 (0.0387) | 0.9994 (0.0387) | 0.8197 (0.5247) | 1.2197 (0.1406) |
| | 1.4954 (0.2942) | 1.9877 (0.0229) | 0.4765 (1.3416) | 2.1102 (0.6665) | 0.3701 (2.6793) | 2.7015 (0.5149) | 0.8207 (0.7555) | 1.2213 (0.3718) | 0.7092 (1.6888) | 1.4099 (0.3709) |

**Table 3: Bayes Estimates and MSE (in Parenthesis) for $\alpha$ and $\beta$ using Inverse Levy Prior** $(r_1 = r_2 = 0.1, 0.5 \& 0.9)$

| n | $\hat{\alpha}_{SELF}$ | $\hat{\beta}_{SELF}$ | $\hat{\alpha}_{LLF}$ | | $\hat{\beta}_{LLF}$ | | $\hat{\alpha}_{GELF}$ | | $\hat{\beta}_{GELF}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 25 | 0.5059 (0.0233) | 0.9598 (0.2144) | 0.7774 (0.1003) | 1.2894 (0.6465) | 0.6181 (0.3586) | 1.6161 (0.5924) | 1.4269 (0.8825) | 0.7078 (0.0665) | 1.0192 (0.2131) | 0.9802 (0.2131) |
| | 0.9995 (0.2362) | 1.4568 (0.0605) | 0.6096 (0.3885) | 1.6566 (0.6673) | 0.4818 (1.0953) | 2.0726 (0.3865) | 1.0152 (0.2364) | 0.9947 (0.2362) | 0.8274 (0.5110) | 1.2078 (0.1440) |
| | 1.4707 (0.2454) | 1.9504 (0.1961) | 0.4845 (1.2756) | 2.1098 (0.6165) | 0.3758 (2.8315) | 2.6549 (0.6226) | 0.8367 (0.6845) | 1.2066 (0.3306) | 0.7146 (1.8458) | 1.3986 (0.5553) |
| 50 | 0.5030 (0.0074) | 0.9761 (0.9416) | 0.7781 (0.0847) | 1.2867 (0.6264) | 0.6135 (1.0903) | 1.6292 (1.3369) | 1.4205 (0.8548) | 0.7074 (0.0504) | 1.0116 (0.9411) | 0.9881 (0.9411) |
| | 0.9998 (0.0779) | 1.4751 (0.1461) | 0.6080 (0.2315) | 1.6527 (0.5040) | 0.4779 (1.1900) | 2.0911 (0.4949) | 1.0075 (0.0780) | 0.9974 (0.0779) | 0.8229 (0.6038) | 1.2148 (0.2268) |
| | 1.4855 (0.0934) | 1.9740 (0.1875) | 0.4784 (1.1368) | 2.1136 (0.4697) | 0.3723 (2.8361) | 2.6841 (0.6549) | 0.8265 (0.5467) | 1.2157 (0.1740) | 0.7113 (1.8474) | 1.4055 (0.5402) |
| 100 | 0.5015 (0.0031) | 0.9895 (0.4392) | 0.7784 (0.0807) | 1.2854 (0.6200) | 0.6096 (0.5914) | 1.6401 (0.8489) | 1.4173 (0.8447) | 0.7073 (0.0461) | 1.0051 (0.4391) | 0.9947 (0.4391) |
| | 0.9999 (0.0387) | 1.4878 (0.0622) | 0.6072 (0.1930) | 1.6507 (0.4622) | 0.4751 (1.1123) | 2.1042 (0.4271) | 1.0037 (0.0387) | 0.9987 (0.0387) | 0.8197 (0.5248) | 1.2198 (0.1405) |
| | 1.4928 (0.2942) | 1.9876 (0.0229) | 0.4753 (1.3440) | 2.1153 (0.6728) | 0.3700 (2.6794) | 2.7017 (0.5152) | 0.8215 (0.7545) | 1.2202 (0.3724) | 0.7092 (1.6888) | 1.4099 (0.3709) |

From the findings of above tables, it can be observed that the large sample distribution could be improved when prior is taken into account. It is observed that within each prior SELF performs better results as compared to other loss functions. Further it is observed that the mean square error based on different priors tends to decrease with the increase in sample size. It implies that the estimators obtained are consistent.

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