A Generalized Log-Weibull Distribution with Bio-Medical Applications

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Abstract: Here we consider a generalized log-transformed version of the Weibull distribution and investigate some of its important properties like expressions for the cumulative distribution function hazard rate function, quantile function, characteristic function, raw moments, incomplete moments, etc. The distribution and moments of order statistics are obtained along with some results on certain structural properties of the distribution. The maximum likelihood estimation of the parameters of the distribution is attempted for both complete and censored data sets and the usefulness of the distribution is illustrated with the help of real-life data sets from biomedical fields.

Keywords: Hazard rate, Maximum likelihood estimation, Model selection, Order statistics, Reliability measures, Censored data.

1. INTRODUCTION

The generalized versions of Weibull distribution (WD) and their applications have gained much importance in fields of engineering, ecology, medicine, pharmacy, etc. due to the flexibility of these distributions in handling survival data. Many such modifications of the WD were considered by various researchers including [1-5], etc. An exponentiated version of the WD having the name 'the exponentiated Weibull distribution' was introduced by [6] to incorporate the bath-tub shape for its hazard rate function through the cumulative distribution function (c.d.f.)

$$G_1(y) = \left\{ 1 - e^{[-\sigma y^\beta]} \right\}^\delta, \tag{1}$$

for any y > 0, with scale parameter σ > 0 and shape parameters β > 0, δ > 0. A distribution with c.d.f. (1) is hereby denoted as EWD(σ, β, δ). Various properties of the distribution were further investigated in detail by several researchers like [7-9], etc. More recently, [10] considered certain properties of a log-transformed version of the one parametric Weibull distribution capable of dealing with truncated data sets using the name, 'the log-Weibull distribution (LWD)' with c.d.f.

$$G_2(y) = 1 - e^{\{-[\ln(y)]^c\}}, \tag{2}$$

for y > 1 and shape parameter c > 0. The distribution with c.d.f. (2) is hereafter referred to as the LWD(c).

In this paper, we consider a generalized version of the LWD(c) using the name 'the generalized log-Weibull distribution (GLWD)'. It is seen that the GLWD possess five distinct shapes for its hazard rate

function including most of the monotone as well as non-monotone hazard rate shapes such as increasing, decreasing, bathtub, upside-down bathtub, and 'S' shapes and is more flexible in terms of its measures of central tendency, dispersion, skewness, and kurtosis, which highlights the utility of the model as a lifetime distribution. As such the proposed distribution can be considered to be a better alternative for many of the recently developed modifications of the WD. The paper is organized as follows: In section 2 we introduce the GLWD and discuss its important properties. In section 3, some structural properties of the distribution are dealt with and Section 4 derives the distribution and moments of the order statistics of the GLWD. Section 5 contains the maximum likelihood estimation of the parameters of the distribution for complete and censored cases. In section 6, the usefulness of the model is illustrated with the help of both complete and censored real-life data sets.

2. GENERALIZED LOG-WEIBULL DISTRIBUTION: DEFINITION AND PROPERTIES

Here we present the definition of the generalized log-Weibull distribution and discuss its important properties.

Definition 2.1 A continuous random variable Y is said to have "the generalized log-Weibull distribution (GLWD)" if its c.d.f. is of the following form, for any α>0, β>0 and y >1.

$$F_Y(y) = \left\{ 1 - e^{-[\ln(y)]^\alpha} \right\}^\beta \tag{3}$$

Henceforth we write GLWD(α, β) for denoting this distribution, it is necessary. When β = 1, the GLWD(α, β) reduces to the LWD(α) with c.d.f (2).

We give the expressions for the probability density function (p.d.f.) $f_Y(y)$, survival function $\bar{F}_Y(y)$, the hazard rate function $h_Y(y)$ and the reversed hazard rate function $\tau_Y(y)$ of the GLWD(α, β) as given below.

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$$f_Y(y) = \frac{\beta\alpha}{y}[\ln(y)]^{\alpha-1}e^{-[\ln(y)]^\alpha}\left[1 - e^{-[\ln(y)]^\alpha}\right]^{\beta-1}, \tag{4}$$

$$\overline{F}_Y(y) = 1 - \left\{1 - e^{-[\ln(y)]^\alpha}\right\}^\beta,$$

$$h_Y(y) = \frac{\alpha\beta}{y}\left\{1 - e^{-[\ln(y)]^\alpha}\right\}^{\beta-1}e^{-[\ln(y)]^\alpha}[\ln(y)]^{(\alpha-1)}\left\{1 - \left[1 - e^{-[\ln(y)]^\alpha}\right]^\beta\right\}^{-1} \tag{5}$$

and

$$\tau_Y(y) = \frac{\alpha\beta}{y}\left\{1 - e^{-[\ln(y)]^\alpha}\right\}^{-1}e^{-[\ln(y)]^\alpha}[\ln(y)]^{(\alpha-1)}. \tag{6}$$

A practical interpretation of the $\mathrm{GLWD}\,(\alpha,\beta)$ can be provided whenever β is an integer. Consider a device that is constituted of independent and identically distributed components having the LWD(α) with c.d.f. (3), connected in a parallel system so that the device fails only if the last component in the system fails. Let $Y_1, Y_2, \ldots, Y_\beta$ denote the lifetimes of the components and let Y be the life of the system with c.d.f. $F_Y(y)$. Then

$$F_Y(y) = P(Y \le y) = P(Y_1 \le y, Y_2 \le y, \ldots, Y_\beta \le y)$$
$$= [P(Y_1 \le y)]^\beta = [G_2(y)]^\beta.$$

Clearly, in the light of (3), the lifetime of the device has the GLWD(α, β).

The plots of the p.d.f., the c.d.f. and the hazard rate function of the $\mathrm{GLWD}(\alpha,\beta)$ for particular values of its parameters are presented in Figures **1**, **2**, and **3**

respectively. Based on Figures **1**, **2**, and **3** we have the following observations regarding the shapes of the c.d.f., p.d.f. and hazard rate function of the $\mathrm{GLWD}(\alpha,\beta)$

- From Figure **1** it can be observed that the p.d.f. of the $\mathrm{GLWD}(\alpha,\beta)$ is a decreasing function for values of α and β such that αβ< 1 whereas the p.d.f. is unimodal when αβ> 1.

- Figure **2** reveals that the c.d.f. F_Y (.) of the $\mathrm{GLWD}(\alpha,\beta)$ coincides at the point (e, 0.63212$^\beta$) for fixed values of the parameter β and various values of α.

- From Figure **3**, it is clear that the hazard rate function takes five different shapes including decreasing, increasing, bathtub shape, S-shape, and upside-down bathtub shape depending on the values of the parameters α and β, which illustrates the flexibility of the distribution to model data sets.

On inverting the c.d.f. $F_Y(y_p)$ of the $\mathrm{GLWD}(\alpha,\beta)$, we obtain the expression for the quantile function y_p in which p is a Uniform(0,1) random variable as

$$y_p = e^{\left\{\left[-\ln\left(1-p^{\beta^{-1}}\right)\right]^{\alpha^{-1}}\right\}}. \tag{7}$$

On substituting p = 0.5 in (7), we obtain the median (M) of the $\mathrm{GLWD}(\alpha,\beta)$ as

(a)

(b)

(c)

(d)
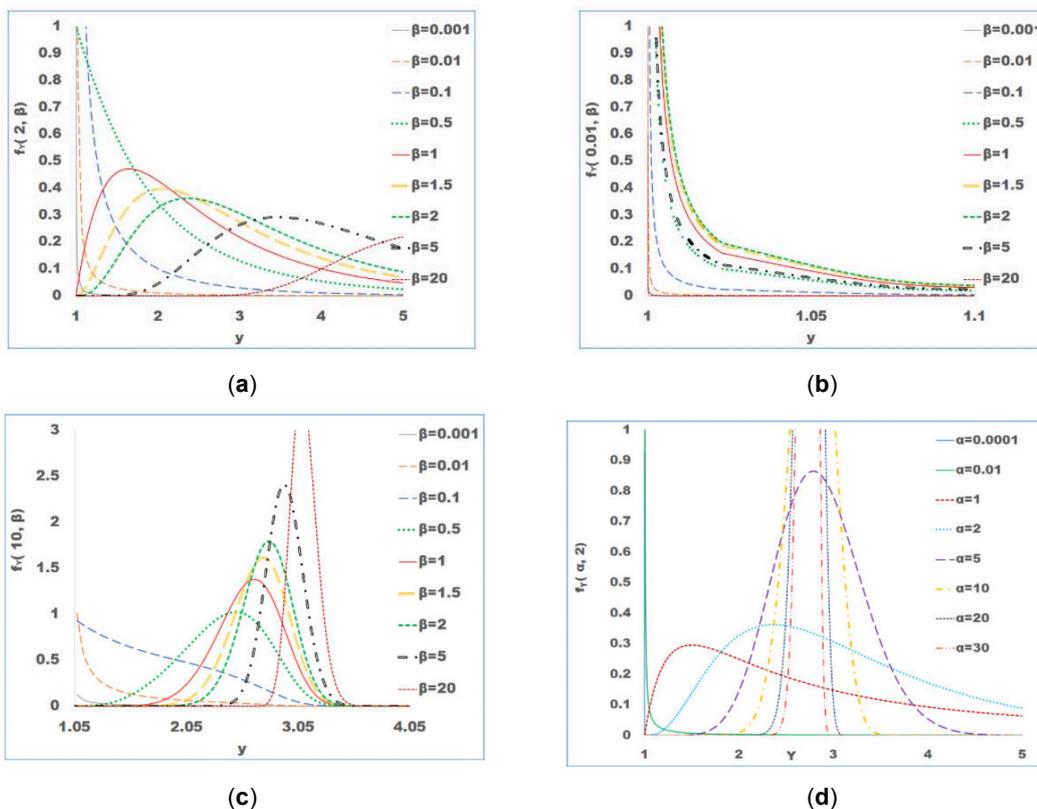
Figure 1: Plots of the p.d.f. of the GLWD for particular values of its parameters.

(**a**) Shapes of the p.d.f. for *α*= 0.01 and particular values of *β*. (**b**) Shapes of the p.d.f. for *α*= 2 and particular values of *β*. (**c**) Shapes of the p.d.f. for *α*= 10 and particular values of *β*. (**d**) Shapes of the p.d.f. for *β*= 2 and particular values of *α*.

(**a**)   (**b**)
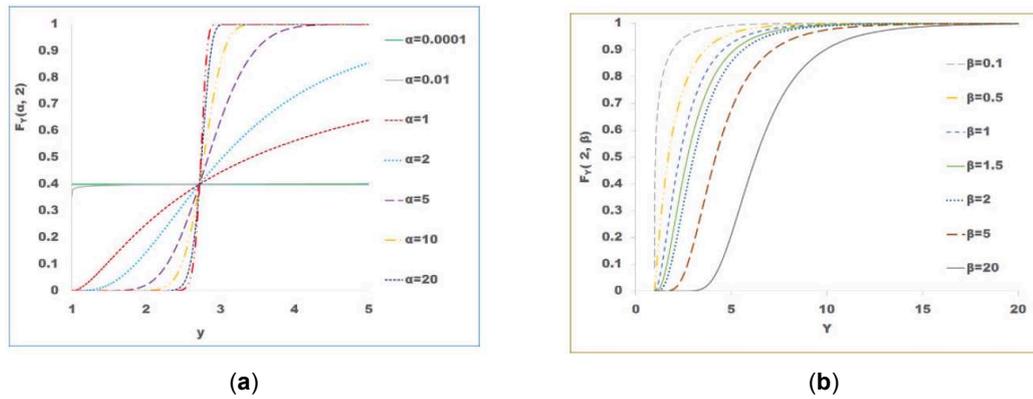
Figure 2: Plots of the c.d.f. of the GLWD for particular values of its parameters.

(**a**) Shapes of the c.d.f. for *β*= 2 and particular values of *α*. (**b**) Shapes of the c.d.f. for *α*= 2 and particular values of *β*.

(**a**)   (**b**)

(**c**)   (**d**)
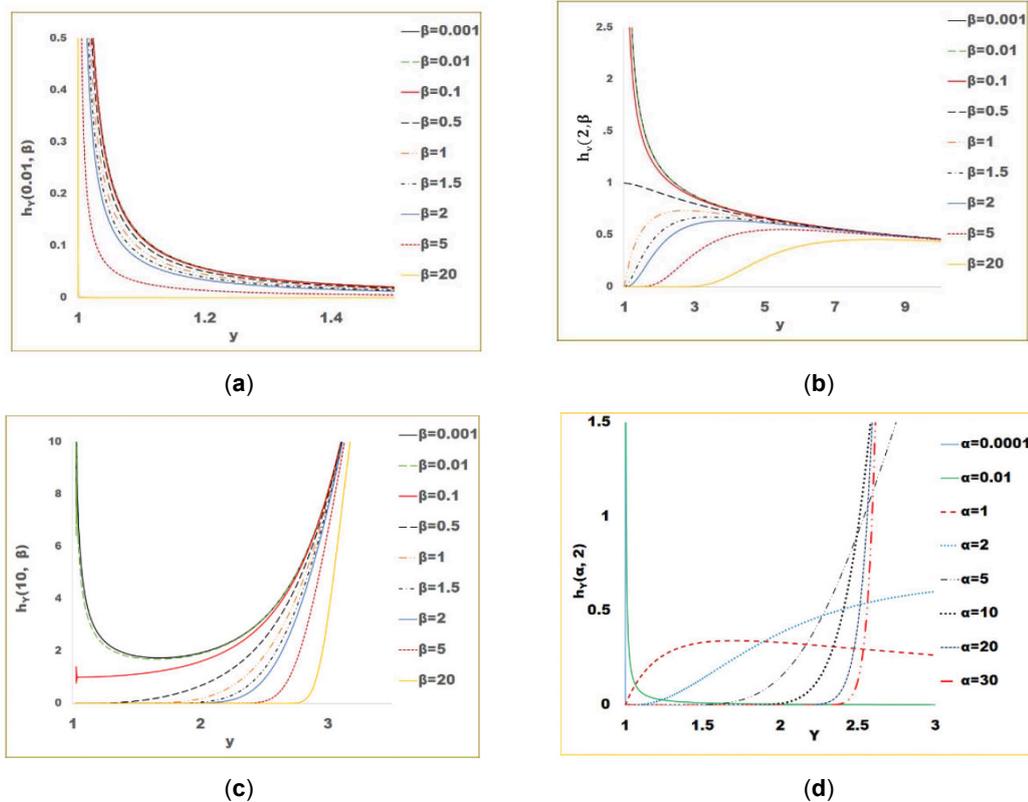
Figure 3: Plots of the hazard rate function of the GLWD for particular values of its parameters.

(**a**) Shapes of the hazard rate function for *α*=0.01 particular values of *β*. (**b**) Shapes of the hazard rate function for *α*=2 and particular values of *β*. (**c**) Shapes of the hazard rate function for *α*=10 particular values of *β*. (**d**) Shapes of the hazard rate function for *β* = 2 and particular values of *α*.

$$\text{M} = e^{\left\{\left[-\ln\left(1-0.5^{\beta-1}\right)\right]^{\alpha-1}\right\}}.$$

Also on differentiating the p.d.f. (4) for y, we have

$$f'_Y(y) =$$
$$f_Y(y)y^{-1}\left\{[\ln(y)]^{-1}\left[\alpha\left(1-[\ln(y)]^\alpha\frac{1-\beta e^{-[\ln(y)]^\alpha}}{1-e^{-[\ln(y)]^\alpha}}\right)-1\right]-1\right\}. \tag{8}$$

The mode of the $\text{GLWD}(\alpha,\beta)$ is obtained from (8), as the solution of the equation $f'_Y(y) = 0$, which reduces to

$$(\alpha-1)[\ln(y)]^{-1} - \alpha[\ln(y)]^{\alpha-1} +$$
$$\alpha(\beta-1)[\ln(y)]^{\alpha-1}\left\{e^{[\ln(y)]^\alpha}-1\right\} - 1 = 0. \tag{9}$$

Using the condition for unimodality, it can be observed that the $\text{GLWD}(\alpha,\beta)$ is unimodal if $\frac{d^2f'_Y(y)}{dy^2} <$ 0, which on simplification reduces to

$$\left[1-\frac{\alpha-1}{[\ln(y)]}\right]\left[\frac{1-\beta e^{-[\ln(y)]^\alpha}}{1-e^{-[\ln(y)]^\alpha}}\right] + 1 \leq$$
$$\frac{\alpha}{[\ln(y)]}\left[\alpha(d-1)\frac{[\ln(y)]^{2\alpha-1}e^{-[\ln(y)]^\alpha}}{[1-e^{-[\ln(y)]^\alpha}]^2} + \frac{\alpha-1}{\alpha}\left[\frac{1}{[\ln(y)]}-1\right]\right].$$

The following observations can be made on the values of the modes of the GLWD(α, β) for various values of α and β.

- When β= 1 and α<1 such that $\alpha\beta$<1.

  In this case, the p.d.f. of the $\text{GLWD}(\alpha, \beta)$ is a decreasing function of y.

- When β= 1 and α>1 such that $\alpha\beta$>1.

  In this case the p.d.f. is unimodal in shape and the mode (M_o) is obtained from (9)

  as the solution of the equation $[\ln(M_o)]^\alpha = 1 - \frac{[\ln(M_o)]+1}{\alpha}$.

- When α= 1 and β>1 such that $\alpha\beta$>1.

  Here, the p.d.f. of the $\text{GLWD}(\alpha, \beta)$ is unimodal in shape and the mode is obtained

  from (9) as $M_o = \left[\frac{\beta+1}{\beta-1}\right]$.

- When α= 1, β<1 or α<1, β<1 such that $\alpha\beta$<1.

  In both these cases, the p.d.f. is a decreasing function of y.

- When α<1 and β>1 such that $\alpha\beta$>1.

  In this case, the p.d.f. of the $\text{GLWD}(\alpha, \beta)$ is unimodal and M_o is obtained from (9) as the solution of the equation $[\ln(M_o)]\left\{1 + \frac{\alpha(\beta+1)}{2}[\ln(M_o)]^\alpha\right\} = \alpha\beta - 1$.

Moreover, differentiating the reversed hazard rate function (6) concerning y, we have

$$\tau'_Y(y) = \tau_Y(y)y^{-1}\left\{(\alpha-1)[\ln(y)]^{-1} - \alpha\frac{[\ln(y)]^{\alpha-1}}{1-e^{-[\ln(y)]^\alpha}} - 1\right\}. \quad (10)$$

Based on (10) we have the following result on the log-concavity of the c.d.f. of the $\text{GLWD}(\alpha, \beta)$, the proof of which is straightforward and hence omitted.

Result 2.1 The c.d.f. $F_Y(y)$ of the GLWD(α, β) is log-concave if $\frac{[ln(y)]^\alpha}{1-e^{[-(ln(y))^\alpha]}} > 1 - \frac{[ln(y)]+1}{\alpha}$.
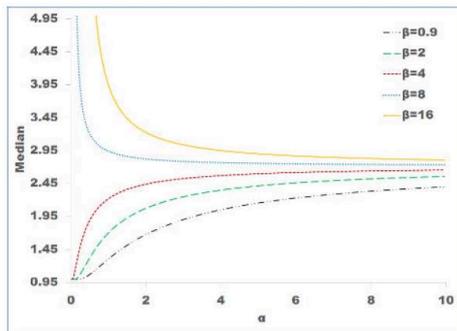
We have plotted the values of the median and mode of the $\text{GLWD}(\alpha, \beta)$ for arbitrary values of β and particular values of the parameter α (Figure **4**).

Now we present certain series representations and integrals which we require in the sequel for deriving expressions for the characteristic function and moments of the $\text{GLWD}(\alpha, \beta)$.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}B(j,k) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}B(j,k-j) \quad (11)$$

Also, for any a $\in\mathcal{R}$,

$$(1+x)^a = \sum_{j=0}^{\infty}(a+1-j)_j\frac{x^j}{j!}, \quad (12)$$

in which $(z)_k = z(z+1)\dots(z+k-1)$, for k $\geq$ 1 with $(z)_0 = 1$.

The incomplete Gamma function $\gamma(\lambda, z)$ is defined as

$$\int_0^u z^{v-1}e^{-\lambda z}\,dz = \lambda^{-v}\gamma(v, \lambda u). \quad (13)$$

The expressions of the characteristic function and the r$^{\text{th}}$ raw moment of the $\text{GLWD}(\alpha, \beta)$ are obtained through the following results, the proofs of which are provided in Appendix A.

Result 2.2 The characteristic function of the GLWD(α, β) is

$$\Phi_Y(t) = \beta\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(\beta-m)_m(-1)^m(kb)^{j-m}e^{ak}\Gamma[\alpha^{-1}(j-m)+1]}{m!(j-m)!(m+1)^{[\alpha^{-1}(j-m)+1]}}\frac{(it)^k}{k!}, \quad (14)$$

when Re[$\{\alpha^{-1}(j-m)+1\}$] > 0, t $\in \mathcal{R}$ and $i = \sqrt{-1}$.

Remark 2.1 As a special case, when α = 2, using equations (3.462) and (9.254) in [11], the characteristic function of the $GLWD(2, \beta)$ for $D_{-2}(a) = -\sqrt{\pi/2}e^{\left(\frac{a^2}{4}\right)}\left[\sqrt{\pi/2}e^{\left(-\frac{a^2}{2}\right)} - a\left(1 - \phi'(a/\sqrt{2})\right)\right]$ is

$$\Phi_Y(t) = \begin{cases} \sum_{k=0}^{\infty}\left[\frac{(it)^k}{k!}\sum_{i=0}^{\beta-1}\binom{\beta-1}{i}\frac{(-1)^i}{i+1}e^{\left[\frac{(it)^2}{8(i+1)}\right]}D_{-2}\left[\frac{-(it)}{\sqrt{2}(i+1)}\right]\right], if\ \beta \in N \\ \sum_{k=0}^{\infty}\left[\frac{(it)^k}{k!}\sum_{i=0}^{\infty}\frac{(\beta-i)_i}{i!}\frac{(-1)^i}{i+1}e^{\left[\frac{(it)^2}{8(i+1)}\right]}D_{-2}\left[\frac{-(it)}{\sqrt{2}(i+1)}\right]\right], if\ \beta \notin N \end{cases}$$

(**a**)
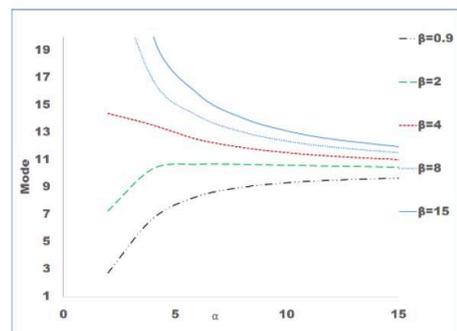
(**b**)

Figure 4: Plots of the median and mode of the GLWD for particular values of its parameters.

(**a**) Median of the GLWD for particular values of α and β. (**b**) Mode of the GLWD for particular values of α and β.

in which φ (.) is the standardized normal distribution function.

Result 2.3 The rth raw moment of the $GLWD(\alpha, \beta)$ is given by

$$\mu_r(y) = \begin{cases} \beta \sum_{k=0}^{\infty} A_k(\alpha, \beta) \Gamma\left(\frac{k}{\alpha} + 1\right), if\ \beta \in N \\ \beta \sum_{k=0}^{\infty} B_k(\alpha, \beta) \Gamma\left(\frac{k}{\alpha} + 1\right), if\ \beta \notin N, \end{cases} \tag{15}$$

where $A_k(\alpha, \beta) = 1 + \sum_{j=1}^{\beta-1} \frac{(-1)^j \binom{\beta-1}{j}}{(j+1)(k\alpha^{-1}+1)}$, $B_k(\alpha, \beta) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j (\beta-k)_k}{j!(j+1)(k\alpha^{-1}+1)}$ and N is the set of natural numbers.

The expressions for μ $_r$ and $\Phi_Y(t)$ is seen to converge for all values of r $\in$ N as well as the parameters of the GLWD(α, β). It has also been verified that the values of the raw moments of the GLWD can be calculated numerically using statistical packages like MATHEMATICA and MATHCAD. We derive the expression for the rth incomplete raw moment of the GLWD(α, β) through the following result.

Result 2.4 The expression for the rth incomplete raw moment of the GLWD(α, β) can be obtained as $\Delta_r(t) = \beta \sum_{k,j=0}^{\infty} \frac{(r)^k(-1)^j(\beta-j)_j(j+1)^{-\left(\frac{k}{\alpha}+1\right)}}{j!k!} \gamma\left(\left[\frac{k}{\alpha}+1\right], (j+1)(ln(t))^{\alpha}\right),$ (16)

in which γ(λ, z) is the incomplete Gamma function as defined in (13).

Proof. The rth incomplete raw moment $\Delta_r(t)$ is given by

$\Delta_r(t) = E(Y^r/Y < t) = \alpha\beta \int_1^t y^r y^{-1}\{1 - e^{[-[\ln(y)]^{\alpha}]}\}^{\beta-1} e^{[-[\ln(y)]^{\alpha}]}[\ln(y)]^{(\alpha-1)}dy.$ (17)

By using the substitution $z = [ln(y)]^{\alpha}$ in (17), we obtain

$\Delta_r(t) = \sum_{k=0}^{\infty}\left[\frac{(r)^k}{k!}\beta \int_0^{(\ln(t))^{\alpha}} z^{k\alpha^{-1}}e^{-z}(1-e^{-z})^{\beta-1}dz\right].$ (18)

On simplifying (18) in the light of (12) and (13), we obtain (16).

We obtain the expressions for the mean deviations about the mean and the median of the $GLWD(\alpha, \beta)$ with c.d.f. F$_Y$(.), in terms of its rth incomplete moment, $\Delta_r(.)$ can be obtained as

$E(|Y - \mu_1|) = 2\mu_1 FY(\mu_1) - 2\Delta_1(\mu_1)$ and $E(|Y - M|) = M - 2\Delta_1(M)$ respectively, where $\Delta_1(t)$ is as defined in (16), when r =1.

Now we present certain expressions for the percentile measures of skewness and kurtosis of the $GLWD(\alpha, \beta)$. Percentile measures of skewness and kurtosis of distribution are less affected by the tail behaviour of the distribution or by outliers (see [12])

and find use in cases where the moment measures are infinite. Galton's and Bowley's measures of skewness, S$_G$, and S$_B$ are defined as

$$S_G = \frac{y_{0.8} - y_{0.5}}{y_{0.5} - y_{0.2}} \tag{19}$$

and

$$S_B = \frac{(y_{0.75}-y_{0.5})-(y_{0.5}-y_{0.25})}{y_{0.75}-y_{0.25}}, \tag{20}$$

while the expression for the Schmid - Trede measure of kurtosis L is

$$L = \frac{y_{0.975} - y_{0.025}}{y_{0.75} - y_{0.25}}, \tag{21}$$

in which y$_p$ is the pth quantile of the $GLWD(\alpha, \beta)$ as given in (7). The percentile measures of skewness and kurtosis of the GLWD(α, β) are obtained through the following results, the proofs of which follow directly from (19), (20), and (21) respectively, in the light of (9). We have also calculated the values of S$_G$, S$_B$, and L of the $GLWD(\alpha, \beta)$ for particular values of its parameters and plotted them in Figure **5**.

Result 2.5 The Galton's and Bowley's percentile measures of skewness, denoted by S$_G$ and S$_B$ respectively, of the GLWD are given by $S_G = \frac{e^{(\eta_{0.8}-\eta_{0.5})}-1}{1-e^{(\eta_{0.2}-\eta_{0.5})}}$ and $S_B = \frac{\{e^{(\eta_{0.75}-\eta_{0.5})}+e^{(\eta_{0.25}-\eta_{0.5})}-2\}}{e^{(\eta_{0.75}-\eta_{0.5})}-e^{(\eta_{0.25}-\eta_{0.5})}}$, in which $\eta_p = \left[-ln\left(1-p^{\beta^{-1}}\right)\right]^{\alpha^{-1}}$, for 0 < p < 1.

Result 2.6 For the $GLWD(\alpha, \beta)$, the Schmid-Trede percentile measure of kurtosis (L) is given by $L = \frac{e^{\eta_{0.975}}-e^{\eta_{0.025}}}{e^{\eta_{0.75}}-e^{\eta_{0.25}}}$, where η$_p$ is as defined in Result 2.5.

In the light of Results 2.5 and 2.6, we have the following observations.

- The GLWD is symmetric when $e^{\eta_{0.8}} + e^{\eta_{0.2}} = 2e^{\eta_{0.5}}$ and is negatively (or positively) skewed if $e^{\eta_{0.8}} + e^{\eta_{0.2}}$ is less than (or greater than) $2e^{\eta_{0.5}}$.

- The GLWD is mesokurtic when $e^{\eta_{0.975}} - e^{\eta_{0.025}} = 2.9058(e^{\eta_{0.75}} - e^{\eta_{0.25}})$ and platykurtic (or leptokurtic) when $e^{\eta_{0.975}} - e^{\eta_{0.025}}$ is less than (or greater than) $2.9058(e^{\eta_{0.75}} - e^{\eta_{0.25}})$.

The following result gives an expression for the geometric mean (GM) of the $GLWD(\alpha, \beta)$, the proof of which follows directly from the definition of the GM as ln(GM)=E[ln(Y)].

Result 2.7 The GM of the $GLWD(\alpha, \beta)$ is

$ln(GM) = \begin{cases} \beta\Gamma\left[1+\frac{1}{\alpha}\right]\sum_{k=0}^{\beta-1}\binom{\beta-1}{k}(-1)^k(k+1)^{\frac{-1}{\alpha}-1}, if\ \beta \in N \\ \beta\Gamma\left[1+\frac{1}{\alpha}\right]\sum_{k=0}^{\infty}(\beta-k)_k(-1)^k(k+1)^{\frac{-1}{\alpha}-1}, if \beta \notin N. \end{cases}$
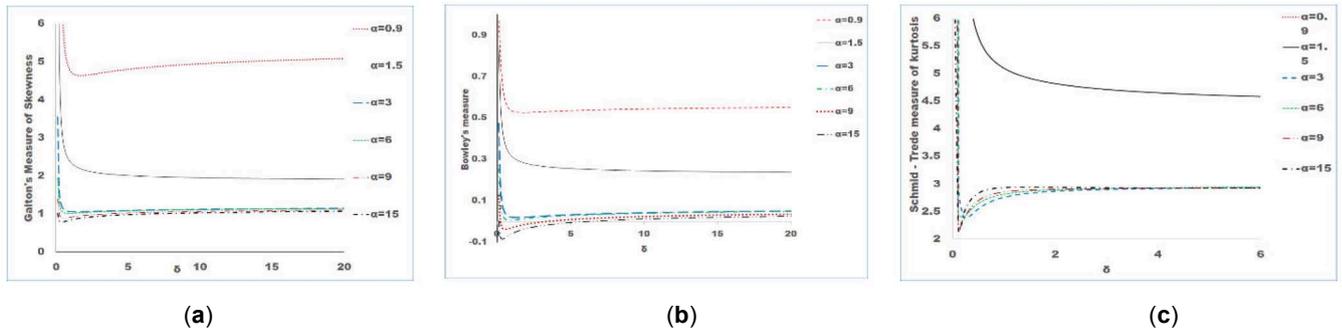
Figure 5: Plots of the percentile measures of skewness and kurtosis of the GLWD for arbitrary values of its parameters.

(**a**) Galton's measure of skewness. (**b**) Bowley's measure of Skewness. (**c**) Schmid-Trede measure of Kurtosis.

The stress-strength reliability concept, initially considered by [13] is used for describing the life of a component having strength Y_2, subjected to a stress Y_1, where both Y_1 and Y_2 are random variables. The component fails if $Y1 > Y2$ and will survive otherwise. The stress-strength reliability measure R, defined as

$$R = P(Y_1 < Y_2) \qquad (22)$$

is the probability that a randomly selected device functions successfully and is a measure of component reliability, having significant applications in areas of engineering, genetics, psychology, physics, and economics. We obtain expressions for the stress-strength reliability measure R of the $\mathrm{GLWD}(\alpha, \beta)$ for fixed values of its parameters through the following results.

Result 2.8 Let Y_i be a random variable following the GLWD(α_i, β_i), for i=1,2; with p.d.f. $f_Y(.)$ as defined in (4). Then the stress-strength reliability measure R is the following, in which β(p, q) is the Beta function.

$$R = \beta_2 \sum_{k=0}^{\infty} (\beta_1 - k + 1)_k \frac{(-1)^k}{k!} \beta(\alpha_1 \alpha_2^{-1} k + 1, \beta_2) \qquad (23)$$

Proof. By the definition of R as given in (22),

$$R = \int_1^{\infty} \left(\int_1^{y_2} f_1(y_1) dy_1 \right) f_2(y_2)\, dy_2$$

$$= \beta_2 \alpha_2 \int_1^{\infty} y_2^{-1} \left\{ 1 - e^{[-[\ln(y_2)]^{\alpha_1}]} \right\}^{\beta_1} \left\{ 1 - e^{[-[\ln(y_2)]^{\alpha_2}]} \right\}^{\beta_2-1} e^{[-[\ln(y_2)]^{\alpha_2}]} [\ln(y_2)]^{(\alpha_2-1)} dy_2. \qquad (24)$$

Integrating (24) using the substitution $z = e^{-\ln(y_2)^{\alpha_2}}$, we get (23) in the light of (12).

Result 2.9 When Y_i follows the GLWD(α, β_i), for i=1,2; with p.d.f. $f_Y(.)$ as defined in (4), the system reliability $R = \frac{\beta_2}{\beta_1 + \beta_2}$.

Proof. By definition,

$$R = \int_1^{\infty} \left(\int_1^{y_2} f_1(y_1) dy_1 \right) f_2(y_2) dy_2$$

$$= \beta_2 \alpha_2 \int_1^{\infty} y_2^{-1} \left\{ 1 - e^{[-[\ln(y_2)]^{\alpha_1}]} \right\}^{\beta_1+\beta_2-1} e^{[-[\ln(y_2)]^{\alpha_2}]} [\ln(y_2)]^{(\alpha_2-1)} dy_2$$

$$= \frac{\beta_2}{\beta_1 + \beta_2}.$$

From Result 2.8 and Result 2.9, it can be observed that the system reliability between two variables following the GLWD depends only on the values of those parameters that vary between the two variables.

We derive an expression for the mean residual life (MRL) function of the $\mathrm{GLWD}(\alpha, \beta)$ through the following result.

Result from 2.10 The MRL function M(.) of the GLWD(α, β) with c.d.f. $F_Y(.)$ as defined in (3) and mean μ_1 is the following,

$$M(y) = \frac{1}{\overline{F_Y(y)}} \{\mu_1 - yF_Y(y) + J_1(y)\} - y, \qquad (25)$$

in which for any for y > 1,

$$J_m(y) = \begin{cases} \sum_{j=0}^{\infty} \sum_{k=0}^{\delta} \frac{\beta}{k} (-1)^k \frac{m}{\alpha j!} \gamma \left(\frac{j+1}{c}, (\ln(y))^{\alpha} \right), & \text{if } \beta \in N \\[2mm] \sum_{j,k=0}^{\infty} (\beta + 1 - k)_k (-1)^k \frac{m}{\alpha k! j!} \gamma \left(\frac{j+1}{\alpha}, (\ln(y))^{\alpha} \right), & \text{if } \beta \notin N \end{cases}$$

where $\gamma(a, b)$ is the incomplete gamma function as given in (13).

Proof. By definition,

$$M(y) = \frac{1}{\overline{F_Y(y)}} \int_y^{\infty} (t - y) f_Y(t) dt = \frac{1}{\overline{F_Y(y)}} \left[\int_y^{\infty} t\, f_Y(t) dt \right] - y. \qquad (26)$$

By considering $I_1 = \int_1^{\infty} t\, f_Y(t) = \mu_1$, the mean of the GLWD and $I_2 = \int_1^y t\, f_Y(t) dt$, we can represent the integral $I = \int_y^{\infty} t\, f_Y(t) dt$ as $I_1 - I_2$.

Then, I = $\mu_1 - I_2$.

Now I_2 can be written as $I_2 = y F_Y(y) - \int_1^y F_Y(t) dt = y F_Y(y) - \upsilon_1(y)$, in which $\upsilon_1(y)$ can be evaluated using

the binomial expansion or (12) depending on whether β is an integer or a real number, respectively. Evaluating (26) using expressions for I_1 and I_2 gives (25).

3. SOME STRUCTURAL PROPERTIES

In this section, we present some structural properties of the $GLWD(\alpha, \beta)$ through the following results. The proofs of Results 3.4 to 3.6 are omitted as they are straightforward and can be obtained directly by the method of transformation of variables.

Result 3.1 If Y be any continuous random variable with c.d.f. F_Y (y), F_Y (y) > 0, for every y ∈ (1, ∞), then

$$E\left\{- ln\left[1 - \left(1 - e^{[-[ln(y)]^\alpha]}\right)^\beta\right]/Y > x\right\} = - ln\left[1 - \left(1 - e^{[-[ln(x)]^\alpha]}\right)^\beta\right] + 1,$$ for y ∈ [1, ∞) if and only if Y has the $GLWD(\alpha, \beta)$.

Proof. The proof follows from Theorem 6 ([12], pp. 259) with

$$h(y) = - ln\left[1 - \left(1 - e^{[-[ln(y)]^\alpha]}\right)^\beta\right] \quad \text{and} \quad d = exp\left[\left\{- ln\left(1 - (1 - e^{-1})^{\beta^{-1}}\right)\right\}^{\alpha^{-1}}\right],$$ since h(y) is strictly an increasing differentiable function in [1, ∞] onto [0, ∞) and d is a positive constant with h(d) = 1 so that for y ∈ [1, ∞),

$$F_Y(y) = 1 - e^{\left[\frac{-h(y)}{h(d)}\right]} = \left\{1 - e^{[-[ln(y)]^\alpha]}\right\}^\beta,$$ which is the c.d.f. of the $GLWD(\alpha, \beta)$.

Result 3.2 If Y be any continuous random variable with c.d.f. F_Y (y) for every y ∈ (1, ∞), then

$$E\left\{ln\left[1 - e^{[-[ln(y)]^\alpha]}\right]/Y \le x\right\} = ln\left[1 - e^{[-[ln(x)]^\alpha]}\right] - \frac{1}{\beta},$$ for y ∈ [1, ∞) if and only if Y has the GLWD(α, β).

Proof. The proof follows from Theorem 9 ([12], pp. 264) with $h(y) = ln\left\{1 - e^{[-[ln(y)]^\alpha]}\right\}$ and $d = -\frac{1}{\beta}$ since $E(h(Y)) = -\frac{1}{\beta}$, and $\lim_{y\to\infty} h(y) = 0$ so that for y ∈ [1, ∞),

$$F_Y(y) = e^{\left[\frac{1}{d}[h(\infty)-h(y)]\right]} = \left\{1 - e^{[-[ln(y)]^\alpha]}\right\}^\beta,$$ which is the c.d.f. of $GLWD(\alpha, \beta)$.

Result 3.3 The c.d.f. F_Y (y) of the GLWD(α, β) tends to the c.d.f. of the EWD(1, α, β) for extremely small values of y.

Proof. The c.d.f. F_Y (y) of the $GLWD(\alpha, \beta)$ given in (3) can be written as the following, for y >1.

$$F_Y(y) = \left\{1 - e^{[-ln(y)]^\alpha}\right\}^\beta \qquad (27)$$

Using y = 1 + t for extremely small t > 0 in (27), we get

$$F_Y(t) = \left\{1 - e^{[-ln(1+t)]^\alpha}\right\}^\beta. \qquad (28)$$

On expanding the term ln(1 + t) in (28) and discarding the second term onwards, we obtain

$$F_Y(t) = \left\{1 - e^{(-t)^\alpha}\right\}^\beta,$$ which is the c.d.f. of the EWD(1, α, β), as given in (1).

Result 3.4 For any α > 0 and β > 0, the random variable Y follows the $GLWD(\alpha, \beta)$ if and only if $Z_1^* = ln(y)$ follows the EWD(1, α, β) having c.d.f. (1).

Result 3.5 For any α > 0 and β > 0, the random variable Y follows the $GLWD(\alpha, \beta)$ if and only if $Z_2^* = [ln(y)]^c$ follows the exponentiated exponential distribution, EED(1, β) of [14].

Result 3.6 For any a ∈ (−∞, ∞), b > 0, α> 0 and β> 0, the random variable Y follows the GLWD(α, β) with p.d.f. (4), if and only if $Z_3^* = vY; v > 0, \beta > 0$ follows a four parametric version of the GLWD(α, β) having c.d.f. $F_Y(y) = \left\{1 - e^{-(b^{-1}[ln(y)-a])^\alpha}\right\}^\beta$, with a = ln(v), and b = β.

4. DISTRIBUTION AND MOMENTS OF ORDER STATISTICS

Let $Y_{i:n}$ be the ith order statistic based on a random sample $Y_1, Y_2, ..., Y_n$ of size n of the $GLWD(\alpha, \beta)$, with p.d.f. $f_Y(y) = f_Y(y; \beta)$ as given in (4) and let $\mu_r = \mu_r(\beta)$ be the rth raw moment of the GLWD(α, β) as given in (15). The distribution and moments of $Y_{i:n}$ can now be derived through the following results.

Result 4.1 For y > 1, the p.d.f. of the ith order statistic $Y_{i:n}$ is given by

$$f_{i:n}(y) = \sum_{k=0}^{n-i} v_{n:i:k} f_Y(y; \beta_{ik}^*), \qquad (29)$$

in which $\beta_{ik}^* = \beta(k + i) - 1$ and $v_{n:i:k} = \frac{\binom{n}{i-1}\binom{n-i+1}{k}(-1)^k(n-k-i+1)\beta}{(k+i)-1}.$

Proof. Consider a random sample of size n from an $GLWD(\alpha, \beta)$ with p.d.f. f_Y (y) and c.d.f. F_Y (y). Then the p.d.f. of the ith order statistic $Y_{i:n}$ can be defined as

$$f_{i:n}(y) = \frac{n!}{(i-1)!(n-i)!}[F_Y(y)]^{i-1}[1 - F_Y(y)]^{n-i}f_Y(y). \qquad (30)$$

By applying the binomial theorem in (30), we have

$$f_{i:n}(y) = \frac{n!\binom{n-i}{k}}{(i-1)!\,(n-i)!}[F_Y(y)]^{i-1}f_Y(y)\sum_{k=0}^{n-i}(-1)^k[F_Y(y)]^k$$

$$= \sum_{k=0}^{n-i}\frac{n!\binom{n-i}{k}(-1)^k\alpha\beta}{(i-1)!(n-i)!y}\left\{1 - e^{[-[ln(y)]^\alpha]}\right\}^{\beta(k+i)-1} e^{[-[ln(y)]^\alpha]}[ln(y)]^{(\alpha-1)}. \qquad (31)$$

On further simplification, (31) gives

$$f_{i:n}(y) = \sum_{k=0}^{n-i}\frac{n!\binom{n-i}{k}}{(i-1)!\,(n-i)!}(-1)^k\frac{\beta}{\beta(k+i)-1}f_Y(y; \beta_{ik}^*),$$

which reduces to (29).

As a consequence of Result 4.1, we have the following corollaries.

Corollary 4.1 For y > 1, the p.d.f. of the smallest order statistic $Y_{1:n}$= min(Y_1, Y_2, ...,Y_n) is

$$f_{1:n}(y) = \sum_{k=0}^{n-1} \binom{n}{k+1} f_Y(y; (k+1)\beta).$$

Corollary 4.2 For y > 1, the largest order statistic $Y_{n:n}$= max(Y_1, Y_2, ..., Y_n) has the GLWD(α,nβ) with c.d.f. $F_{n:n}(y) = \left\{1 - e^{-[ln(y)]^\alpha}\right\}^{n\beta}$.

Corollary 4.3 The largest order statistic $Y_{n:n}$ follows the GLWD(c, nβ) if and only if Y_1 follows the GLWD.

Corollary 4.4 For y > 1, the p.d.f. of the median $Y_{m+1:n}$, for a sample of size n = 2m+1, is

$$f_{(m+1:n)}(y)$$
$$= \sum_{k=0}^{m} (-1)^k \binom{2m+1}{m} \binom{m}{k} \frac{(m+1)}{m+k+1} f_Y(y; \beta(m+k+1)).$$

Further, the rth raw moment of the ith order statistic $Y_{i:n}$ of the GLWD(α, β) is provided through the following result, the proof of which follows from Results 2.3 and 4.1.

Result 4.2 The rth raw moment of $Y_{i:n}$ is

$$\mu_{r(i:n)}(y) = \sum_{k=0}^{i-1} v_{n:i:k} \mu_r(\beta_{ik}^*), \qquad (32)$$

in which r > 0 and $v_{n:i:k}$ and β_{ik^*} are as defined in (29).

5. ESTIMATION

In this section, we discuss the maximum likelihood estimation of the parameters of the GLWD(α, β) and derive the likelihood equations for complete and right-censored cases. A data set of observations without any missing value is termed as an uncensored/complete set. The likelihood function for a complete data set having X_1, X_2, ..., X_n is given by

$$L_1(\Theta) = \prod_i f(x_i, \Theta).$$

Censored data is regularly encountered in survival and reliability analysis as the information regarding the survival time of some of the observations understudy may remain incomplete or unknown. According to [15], censored data sets represent a particular type of missing data. Assume that we have a random sample of *n* units with true survival times T_1, T_2, ..., T_n having p.d.f. f(x) and c.d.f. F(x). However, due to right censoring such as staggered entry, loss to follow-up, competing risks (death from other causes) or any combination of these, it might be impossible to observe the survival times in all of these n cases. Thus, a subject can either be observed for its full lifetime or can

be censored. Clearly, the observed data are the minimum of the survival time and censoring time for each unit. Assume that C_1, C_2, ..., C_n are the censoring times of the n units drawn independently of T_i, i = 1, 2, ..., n. On each of n units, we observe n random pairs (X_i, η_i), in which X_i = min(T_i, C_i) and

$$\eta_i = \begin{cases} 1, \text{for } T_i < C_i \\ 0, \text{for } T_i > C_i \end{cases}$$

for i = 1,2,...,n. Clearly η_i, the censorship indicator indicates whether T_i is censored or not. Then, the likelihood function for the censored data set is given by

$$L_2(\Theta) = \prod_{\eta_i=1} f(x_i; \Theta) \prod_{\eta_i=0} \overline{F}(x_i; \Theta). \qquad (33)$$

5.1. Estimation of Parameters for the GLWD for Complete Data Sets

Consider a random sample constituting of Y_1, Y_2, ..., Y_n observations taken from the GLWD(α, β). The log-likelihood function for the vector of parameters Θ = (α,β) is given by

$$l_1(\Theta) =$$
$$\sum_{i=1}^{n} \{\ln(\beta\alpha) - \ln(y_i) + (\beta-1)\ln\{1 - \exp[-[\ln(y_i)]^\alpha]\} - [\ln(y_i)]^\alpha + (\alpha-1)\ln(\ln(y_i))\}. \qquad (34)$$

Differentiating the log-likelihood function (34) with respect to the parameters α and β respectively, and equating to zero, we obtain the following likelihood equations.

$$\frac{dl_1(\Theta)}{d\alpha} = n\alpha^{-1} + \sum_{i=1}^{n} \ln[\ln(y_i)] \{1 - [\ln(y_i)]^\alpha B_\Theta(y_i)\} = 0$$

and

$$\frac{dl_1(\Theta)}{d\beta} = n\beta^{-1} + \sum_{i=1}^{n} \ln[1 - \exp\{-[\ln(y_i)]^\alpha\}] = 0, \quad (35)$$

in which $B_\Theta(y_i) = \frac{1 - \beta \exp[-[\ln(y_i)]^\alpha]}{1 - \exp[-[\ln(y_i)]^\alpha]}$.

5.2. Estimation of Parameters for the GLWD(α, β) for Censored Data Sets

Let r be the number of failures among the n units. Then, using (4) and (3) in (33) the likelihood function of the GLWD(α, β) for censored data set is given by

$$L_2(\Theta) = \prod_{\eta_i=1} \beta\alpha y_i^{-1} [\ln(y_i)]^{\alpha-1} \exp\{-[\ln(y_i)]^\alpha\} \{1 - \exp\{-[\ln(y_i)]^\alpha\}\}^{\beta-1} \prod_{\eta_i=0} [1 - \{1 - \exp[-(\ln(y_i))^\alpha]\}^\beta]. \qquad (36)$$

From (36), we obtain the corresponding log-likelihood function as

$$l_2(\Theta) =$$
$$r\ln(\alpha) + r\ln(\beta) - \sum_{\eta_i=1} \ln(y_i) - \sum_{\eta_i=1} [\ln(y_i)]^\alpha + (\alpha-1)\sum_{\eta_i=1} \ln(\ln(y_i)) + (\beta-1)\sum_{\eta_i=1} \ln\{1 - \exp[-[\ln(y_i)]^\alpha]\} + \sum_{\eta_i=0} \ln\{1 - \{1 - \exp[-[\ln(y_i)]^\alpha]\}^\beta\}. \qquad (37)$$

Differentiating (37) with respect to the parameters α and β, we have the following likelihood equations for censored observations.

$$\frac{dl_2(\Theta)}{d\alpha}$$
$$= \frac{r}{\alpha} + \sum_{\eta_i=1} \ln[\ln(y_i)]\, [1 - [\ln(y_i)]^{\alpha} B_{\Theta}(y_i)]$$
$$- \beta \sum_{\eta_i=0} \frac{\{1 - \exp[-[\ln(y_i)]^{\alpha}]\}^{\beta-1} \exp[-(\ln(x))^{-\alpha}]\,[\ln(y_i)]^{-\alpha} \ln[\ln(x)]}{1 - \{1 - \exp[-[\ln(y_i)]^{\alpha}]\}^{\beta}}$$

and

$$\frac{dl_2(\Theta)}{d\beta}$$
$$= \frac{r}{\beta} + \sum_{\eta_i=1} \ln\{1 - \exp[-[\ln(y_i)]^{\alpha}]\}$$
$$- \sum_{\eta_i=0} \frac{\ln\{1 - \exp[-[\ln(y_i)]^{\alpha}]\}\{1 - \exp[-[\ln(y_i)]^{\alpha}]\}^{\beta}}{1 - \{1 - \exp[-[\ln(y_i)]^{\alpha}]\}^{\beta}}$$

in which $B_{\Theta}(y)$ is as defined in (35).

These likelihood equations may not always provide a unique solution and in such cases the maximum of the likelihood function is obtained in the border of the domain of the parameters. Hence, we have obtained the second order partial derivatives of the log-likelihood function of the GLWD and by using R software it has been verified that the values of the second order partial derivatives are negative for the estimated parametric values for both the distributions.

6. APPLICATIONS

To illustrate the utility of the $GLWD(\alpha, \beta)$ as a survival model, we make use of the following three data sets of which the first and second are complete data sets while the third one is a censored data set. All these three data sets arise from biomedical fields.

Data Set 1: Data on remission times for a group of leukemia patients given the drug 6-MP from [16].
6, 7, 10, 13, 16, 22, 23

Data Set 2: The data on survival of 40 patients suffering from Leukaemia, from the Ministry of Health Hospitals in Saudi Arabia taken from [17].

15, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

Data Set 3: Censored data discussed in [18] and given in [15]. The data consist of death times (in weeks) of patients with cancer of tongue with aneuploid DNA profile. The observations are

1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61*, 65, 67, 70, 72, 73, 74*, 77,79*, 80*, 81*, 87*, 87*, 88*, 89*, 91, 93, 93*, 96, 97*, 100, 101*, 104, 104*, 108*, 109*, 120*, 131*, 150*, 157, 167, 231*, 240*, and 400*, where asterisks denote censored observations.

The maximum likelihood estimates of the parameters of the $GLWD(\alpha, \beta)$ along with the corresponding values of the standard errors and P-values are computed for the three data sets using the *R* software. The estimated values of the parameters of the $GLWD(\alpha, \beta)$ along with the values of the standard errors, calculated values of the statistics, and corresponding P-values are presented in Tables **1**, **2**, and **3** respectively.

The variance-covariance matrix of the estimators of the parameters of the $GLWD(\alpha, \beta)$ corresponding to Data Sets 1, 2, and 3 are respectively

$$\Sigma_{(1)} = \begin{bmatrix} 106.9564 & 0.9441 \\ 0.9441 & 0.0243 \end{bmatrix},$$

$$\Sigma_{(2)} = \begin{bmatrix} 0.0245 & 0.9592 \\ 0.9592 & 107.7790 \end{bmatrix}$$

and

$$\Sigma_{(3)} = \begin{bmatrix} 0.0029 & 0.0883 \\ 0.0883 & 5.1511 \end{bmatrix}.$$

The performances of the $GLWD(\alpha, \beta)$ as a survival model is compared with that of some of its related models like the LWD, LIWD, IWD, and the WD by using certain information criteria like 'the Akaike information criteria (AIC)', 'the Bayesian information criteria (BIC)',

Table 1:   Fitted Values of the GLWD Corresponding to Data Set 1

Parameter	Estimate	Std Error	Calculated Value	P-value
α	1.4717	0.1561	934283	<2.2e− 16
β	28.6396	10.3420	2.7693	0.0056

Table 2:   Fitted Values of the GLWD Corresponding to Data Set 2

Parameter	Estimate	Std Error	Calculated Value	P-value
α	1.0690	0.0122	87.25	<2.2e− 16
β	1659.0	4.1950	395.50	<2.2e− 16

Table 3: Fitted Values of the GLWD Corresponding to Data Set 3

Parameter	Estimate	Std Error	Calculated Value	P-value
α	0.66283	0.05439	12.1847	<2.2e− 16
β	12.0451	2.2696	5.3071	1.1e− 07

'the corrected Akaike information criteria (AICc)' and 'the consistent Akaike information criteria (CAIC)'. The numerical results obtained are summarised in Tables **4**, **5** and **6**. Moreover, to test the significance of the parameter β , we have carried out the generalized likelihood ratio test (GLRT) procedure for testing the hypothesis H_0: β = 1 against the alternative H_1: β ≠ 1 using the three data sets. The results of the test procedure are provided in Table **7**.

From Tables **4**, **5**, and **6**, it can be observed that the $\mathrm{GLWD}(\alpha, \beta)$ gives a relatively better fit to both complete and censored data sets as compared to the other distributions, since the values of AIC, BIC, AICc, and CAIC are minimum. Also, from Table **7**, it is evident that the parameter β is significant for all three data sets.

7. CONCLUSION

Through this paper we have considered a generalization of the log-Weibull distribution studied by [10], using the name the 'generalized log-Weibull distribution (GLWD)'. Some important theoretical properties of the distribution were investigated including expressions for its characteristic function, moments, certain reliability measures as well as the distribution and moments of order statistics. The maximum likelihood estimation of its parameters for complete as well as censored data sets was considered and the utility of the model in survival analysis was illustrated using three real-life data sets. Based on the present study it can be concluded that the proposed model has much more flexibility compared to many existing models and is relatively

Table 4: Fitting Various Distributions to Data Set 1

Model	Estimates	Log-Likelihood	AIC	BIC	AICc	CAIC
GLWD	α= 1.4717 β = 28.6396	-22.7589	49.5178	49.1013	53.5178	51.1013
LIWD	c= 1.4872	-32.4513	66.9027	66.8486	67.7027	67.8486
IWD	c = 0.5345	-33.287	68.575	68.521	69.375	69.521
LWD	c= 0.8601	-34.9802	71.9603	71.9062	72.7603	72.9062
WD	c = 0.3113	-35.7846	73.5692	73.5151	74.3692	74.5151

Table 5: Fitting Various Distributions to Data Set 2

Model	Estimates	Log-Likelihood	AIC	BIC	AICc	CAIC
GLWD	α = 1.0690 β = 1659.0	-316.8007	637.6014	640.9791	637.9257	642.9791
IWD	c = 0.1961	-394.6037	791.2074	792.8962	791.3126	793.8962
WD	c = 0.1165	-407.9905	817.9810	819.6698	818.0862	820.6698
LIWD	c= 0.7016	-419.8682	841.7364	843.4252	841.8416	844.4252
LWD	c = 0.4183	-433.1577	868.3154	870.0042	868.4206	871.0042

Table 6:   Fitting Various Distributions to Data Set 3

Model	Estimates	Log-Likelihood	AIC	BIC	AICc	CAIC
GLWD	α= 0.6628 β = 12.0451	-192.0819	388.163	392.066	388.408	394.066
IWD	c = 0.1836	-230.2072	462.4144	464.3656	462.4944	465.3656
LIWD	c = 0.54727	-232.6817	467.3634	469.3146	467.4434	470.3146
WD	c = 0.1066	-262.2964	526.5928	528.5440	526.6728	529.5440
LWD	c = 0.3201	-264.5926	531.1852	533.1364	531.2652	534.1364

Table 7:   The Values of Log-Likelihood Functions and P-Values Corresponding to Data Sets for Testing H_0: β= 1 against the Alternate Hypothesis H_1: $\beta \neq 1$

	Likelihood value	Λ	df	P value
Data Set 1	-34.9802	24.4426	1	<0.00001
Data Set 2	-433.1577	232.7140	1	<0.00001
Data Set 3	-264.5926	145.0214	1	<0.00001

more suitable in handling censored and complete survival data sets, especially from bio-medical applications.

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APPENDIX A

Proof of Result 2.2.

By definition, the characteristic function of the $\mathrm{GLWD}(\alpha, \beta)$ is

$$\phi_Y(t) = \alpha\beta \int_1^\infty e^{ity} y^{-1}\{1 - e^{-[\ln(y)]^\alpha}\}^{\beta-1} e^{-[\ln(y)]^\alpha}[\ln(y)]^{(\alpha-1)} dy.$$

By using the substitution $z = [\ln(y)]^\alpha$ and hence expanding the exponential term $e^{\left\{ite z^{\alpha-1}\right\}}$, we obtain

$$\phi_Y(t) = \beta \sum_{k=0}^\infty \frac{(it)^k}{k!} \int_0^\infty e^{kz^{\alpha-1}}\{1 - e^{-z}\}^{\beta-1} e^{-z} dz$$

$$= \beta \sum_{k=0}^\infty \frac{(it)^k (k)^j}{j! k!} \int_0^\infty z^{j\alpha-1}\{1 - e^{-z}\}^{\beta-1} e^{-z} dz. \quad (38)$$

On expanding the term $\{1 - e^{-z}\}^{\beta-1}$ in (38) using (12) and thereby integrating with respect to z, we get (14) in the light of (11).

Proof of Result 2.3.

By definition, the rth raw moment of the $\mathrm{GLWD}(\alpha, \beta)$ is

$$\mu_r = \alpha\beta \int_1^\infty y^r y^{-1}\{1 - e^{-[\ln(y)]^\alpha}\}^{\beta-1} e^{-[\ln(y)]^\alpha}[\ln(y)]^{(\alpha-1)} dy \quad (39)$$

By using the substitution $z = ln(y)$ in (39) we get

$$\mu_r = \alpha\beta \int_0^\infty e^{r(z)} z^{\alpha-1} e^{-(z)^\alpha} \left(1 - e^{-(z)^\alpha}\right)^{\beta-1} dz$$

$$= \sum_{k=0}^\infty \frac{r^k}{k!} \alpha\beta \int_0^\infty z^k z^{\alpha-1} e^{-(z)^c} \left(1 - e^{-(z)^\alpha}\right)^{\beta-1} dz,$$

which reduces to (15) using (12) and the Binomial theorem, in the light of the equation: 3.381(4) of [19].

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