Application of Wald Function to OR and AND Fuzzy Operations in No-Data Problems

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Abstract: Subjective qualifiers of Wald's theory of decision functions are the fuzzy events of the subsequent fuzzy set theory. Wald's notion of subjective qualifiers involves applying integral transforms to convert states of nature into fuzzy events. Probabilities of fuzzy events and arithmetic formulas for fuzzy utility function values are readily derived from Wald's integral transforms. We have applied Zadeh's extension principle to Wald's integral transforms and demonstrated that fuzzy mathematics is effective when applied to multiple subjective probability distributions conjoined by OR and AND operations. In this paper, we focus on no-data problems and construct a fuzzy Bayes' theorem for cases in which a membership function and multiple subjective probability distributions conjoined by OR or AND operations are given. In addition, we devise a formulation for the corresponding decision making problem.

Keywords: Subjective qualifier, fuzzy event, membership function, subjective probability distribution, OR-conjunction, AND-conjunction.

1. INTRODUCTION

Wald's theory of decision functions [1] includes subjective qualifiers, a point which has attracted little attention until now. In the context of the fuzzy set theory [2], which was proposed later, these may be thought of as fuzzy events obtained as the integral transforms of risk functions. Probabilities of fuzzy events [3] and arithmetic formulas for fuzzy utility function values [4] are readily derived from Wald's integral transform formulas. Later, Uemura [5-8] applied Zadeh's extension principle to Wald's integral transform formulas and designed a decision making procedure for fuzzy events. In further work, Hori and Tsubaki [9-11] showed that Zadeh's extension principle is equivalent to Wald's integral transform formulas in cases involving subjective probability distributions conjoined by OR and AND operations. Further, they demonstrated the efficacy and appropriateness of fuzzy mathematics. Finally, working with fuzzy events requires membership functions that convert states of nature and subjective probability distributions, which express the probabilities with which given states occur in nature, into fuzzy events. Here we must take care to note that two distributions are required even for no-data problems. In this paper, we focus on no-data problems. We assume that multiple subjective probability distributions and membership functions are given and we formulate a fuzzy Bayes' theorem.

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2. RELATIONSHIP BETWEEN BAYESIAN STATISTICS AND FUZZY MATHEMATICS

In Bayesian statistics, the prior distribution is proportional to the inverse of the standard deviation, as shown in equation (1). On the other hand, fuzzy mathematics employs a discrete approximation wherein proportionality relations within each segment *i* are combined by the OR operation, as indicated in equation (2). For *i*=0 (the no-data problem, σi=σ), equation (2) simply reduces to equation (1)

$$(Bayes) \qquad f \quad \alpha \quad 1/\sigma \tag{1}$$

$$(Fuzzy) \quad or \quad \{fi\,\alpha\,1/\sigma i\}\,(i=1,\cdots,\infty) \tag{2}$$

It might seem obvious that Bayesian statistics and fuzzy mathematics should yield identical results for decision making in the case of the no-data problem, but in fact it has been proved that the two methods are equivalent only for a risk-neutral decision maker. In cases of decision makers with risk aversion or risk proneness, only fuzzy mathematics may be used to derive logically sound decision making rules. The reason for this is that utility functions based on the certainty equivalent method are themselves "fuzzy", and thus decision making rules that use piecewise linear utility functions based on the certainty equivalent method allow the application of fuzzy mathematics (the extension principle of mappings in particular) within each segment.

Note that if we should obtain many priors in the state of nature, OR-conjunction is the max-max product (fuzzy decision making), AND-conjunction is min-max product (game theory), and the probability mixed

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weighting is same as with Wald-W method (expected utility).

3. OR-CONJUNCTION FUZZY BAYES' THEOREM

We assume that a subjective probability distribution Π(s) has been transformed from a state of nature *s* into a fuzzy event *F* by means of a membership function $\mu_{Fi}(s)$. Using Zadeh's extension principle, we obtain the following equation:

$$\int \mu_{F_i}(s)/\Pi(s) \triangleq \max_{\{x=\Pi(s)\}} \mu_{F_i}(s) = \mu_{F_i}\left(\Pi^{-1}(x)\right) \quad (3)$$

The subjective probability distribution itself may be identified using the lottery method of Savage [12]. For a bell-shaped distribution, this has been shown to be equivalent to Bayesian statistics [13]. Because we are here considering multiple subjective probability distributions conjoined by OR operations, the consolidated distribution obtained from the theory of subjective probability distributions may be defined as follows:

$$\text{Or}\ \{x=\Pi(s)\} \triangleq \max_{\{x=\Pi(s)\}} \mu_{F_i}(s) = \mu_{F_i}\left(\Pi^{-1}(x)\right) \quad (4)$$

Equations (3) and (4) are equivalent and constitute the fuzzy Bayes' theorem for the case of the OR-conjunction.

4. AND-CONJUNCTION FUZZY BAYES' THEOREM

By considering the dual of the case of OR-conjunction, we obtain the following equations:

$$\int \mu_{F_i}(s)/\Pi(s) \triangleq \min_{\{x=\Pi(s)\}} \mu_{F_i}(s) = 1 - \mu_{F_i}\left(\Pi^{-1}(x)\right) \quad (5)$$

$$\text{AND}\ \{x=\Pi(s)\} \triangleq \min_{\{x=\Pi(s)\}} \mu_{F_i}(s) = 1 - \mu_{F_i}\left(\Pi^{-1}(x)\right) \quad (6)$$

Equations (5) and (6) are equivalent and constitute the fuzzy Bayes' theorem for the case of AND-conjunction.

5. APPLICATION TO DECISION-MAKING PROBLEMS

For the case of the OR-conjunction, we introduce a utility function $U(s \mid d)$; for the case of the AND-conjunction, we introduce a loss function $L(s \mid d)$. Here d denotes an action.

5.1. Fuzzy Utility Functions

Equation (7) which follows is a result derived from Zadeh's extension principle, while equation (8) is derived from the theory of utility functions. It is important to note that equations (7) and (8) are equivalent.

$$\int \mu_{F_i}(s)/U(s \mid d) \triangleq \max_{\{x=U(s \mid d)\}} \mu_{F_i}(s) = \mu_{F_i}\left(U^{-1}(x \mid d)\right) \quad (7)$$

$$\text{Or}\ \{x=U(s \mid d)\} \triangleq \max_{\{x=U(s \mid d)\}} \mu_{F_i}(s) = \mu_{F_i}\left(U^{-1}(x \mid d)\right) \quad (8)$$

5.2. Fuzzy Loss Functions

By considering the dual of the case of OR-conjunction, we obtain equations (9) and (10) below.

$$\int \mu_{F_i}(s)/L(s \mid d) \triangleq \min_{\{x=L(s \mid d)\}} \mu_{F_i}(s) = 1 - \mu_{F_i}\left(L^{-1}(x \mid d)\right) \quad (9)$$

$$\text{AND}\{x=L(s \mid d)\} \triangleq \min_{\{x=L(s \mid d)\}} \mu_{Fi}(s) = 1 - \mu_{Fi}(L_{-1}(x \mid d)) \quad (10)$$

5.3. Max-Product Operation and Optimal Action

For the case of OR-conjunction, we multiply the fuzzy utility function by equation (3) or (4) to obtain measures for a maximization problem. Among these measures, that which has the maximum value is taken as the optimal action. For the case of AND-conjunction, we multiply the fuzzy loss function by equation (5) or (6) to obtain measures for a maximization problem. Among these measures, that which possesses the minimum value is taken as the optimal action.

6. EXAMPLE: DECIDING PASSING OR FAILING GRADES IN AN EXAMINATION FOR RECOMMENDATION-BASED ADMISSIONS

As a sample application, we consider the problem of granting or denying a passing score in an examination for recommendation-based university admissions. We assume that the school report on students' grades and conduct written by Japanese schools has the maximum score of 100 points. The actions we consider are "pass the student" and "fail the student". For cases in which the decision is difficult, we add the action "postpone the decision pending an interview". We also assume that the prior probability distribution of exam scores is known (from experience with past exams) and is given in the form of equation (11).

$$\Pi(s) = \exp - \frac{(s-p)^2}{o^2} \quad (11)$$

In addition, we assume the decision maker to be risk-neutral regarding the question of passing or failing the student. In this case, the utility function for the exam score is a linear function with upper and lower bounds. Consequently, by the method of certainty equivalents, the action "pass the student" should be estimated from upper and lower bounds consisting of the four points {(Pi,0), (Ri,1)} (i=1,2). Similarly, the action "fail the student" should be estimated from the four points {(ki,1), (li,0)} (i=1,2).

Then the type 2 utility functions may be identified as equations (12) and (13).

$$U_1(s) = \frac{1}{(k_i - l_i)} s - \frac{l_i}{(k_i - l_i)} = a_i s + b_i \quad (i = 1,2) \tag{12}$$

$$U_2(s) = \frac{1}{R_i - P_i} s - \frac{P_i}{R_i - P_i} = c_i s + d_i \quad (i = 1,2) \tag{13}$$

Here we attempt decision-making in an ambiguous environment. We assume that the school report on students' grades and conduct written by Japanese Schools for student applicants states only "good" or "bad". This designation of "good" or "bad" is a fuzzy event in the score. We assume that the decision maker uses equation (14) as a probability distribution for the exam score, drawing on past experience to estimate the center and width of the distribution subjectively.

$$\mu_{F_1}(s) = \exp - \frac{(s - n_1)^2}{m_1^2}, \mu_{F_2}(s) = \exp - \frac{(s - n_2)^2}{m_2^2} \tag{14}$$

In this case, because the decision maker is risk-neutral and because the prior probability distribution and the probability distributions of the upper and lower bounds for the fuzzy events are all bell-shaped, the utility functions of the fuzzy events have upper and lower bounds for the actions "pass the student" and "fail the student" with the following probability distributions, respectively:

$$\mu_{F_1}(U_1^{-1}(s)) = \exp - \frac{(z - (b_i + a_i n_1))^2}{m_1^2 a_i^2} \quad (i = 1,2) \tag{15}$$

$$\mu_{F_1}(U_2^{-1}(s)) = \exp - \frac{(z - (d_i + c_i n_1))^2}{m_1^2 c_i^2} \quad (i = 1,2) \tag{16}$$

$$\mu_{F_2}(U_1^{-1}(s)) = \exp - \frac{(z - (b_i + a_i n_2))^2}{m_2^2 a_i^2} \quad (i = 1,2) \tag{17}$$

$$\mu_{F_2}(U_2^{-1}(s)) = \exp - \frac{(z - (d_i + c_i n_2))^2}{m_2^2 c_i^2} \quad (i = 1,2) \tag{18}$$

Next, proceeding similarly, we obtain the prior probability distributions for the fuzzy events as equation (19) for the action "pass the student" and equation (20) for the action "fail the student".

$$\Pi(U_1^{-1}(s)) = \exp - \frac{(z - (b_i + a_i p))^2}{o^2 a_i^2} \quad (i = 1,2) \tag{19}$$

$$\Pi(U_2^{-1}(s)) = \exp - \frac{(z - (d_i + c_i p))^2}{o^2 c_i^2} \quad (i = 1,2) \tag{20}$$

Finally, using the max-product operation, the probabilities for the fuzzy events "good" and "bad" can be obtained as equations (21) and (22), respectively. We compare the sizes of these probability values and choose whichever is larger.

$$\begin{aligned}\Pi_i(F_1) &= \sum_{j=1}^{2} \mu(U_j^{-1}(s)) \times \Pi(U_j^{-1}(s)) \quad (i = 1,2) \\ &= \frac{c_i^2 (b_i + a_i n_1) \times (b_i + a_i p) + a_i^2 (d_i + c_i n_1) \times (d_i + c_i p)}{m_1 a_i^2 c_i^2 o}\end{aligned} \tag{21}$$

$$\Pi_i(F_2) = \frac{c_i^2 (b_i + a_i n_2) \times (b_i + a_i p) + a_i^2 (d_i + c_i n_2) \times (d_i + c_i p)}{m_2 a_i^2 c_i^2 o} \tag{22}$$

However, when the sizes of the probabilities for the upper and lower bounds are interchanged, we opt for the action "postpone the decision pending an interview". These rules for decision makers are specified below.

$Case1: \Pi_i(F_1) \geq \Pi_i(F_2)$

Since $m_1, m_2 \geq 0$,

$(m_2 - m_1)\left\{c_i^2(b_i + a_i n_1) \times (b_i + a_i p) + a_i^2(d_i + c_i n_1) \times (d_i + c_i p)\right\} \geq 0 \ (i = 1,2)$

$D^* = D_1$

$Case2: \Pi_i(F_1) \leq \Pi_i(F_2) \ (i = 1,2)$

$D^* = D_2$

$Case3: \Pi_1(F_1) \geq \Pi_1(F_2), \Pi_2(F_1) \leq \Pi_2(F_2)$ or

$\Pi_1(F_1) \leq \Pi_1(F_2), \Pi_2(F_1) \geq \Pi_2(F_2)$

$D^* = \{indifference\}$

7. CONCLUSIONS

Membership functions, which regulate fuzzy events, transform states of nature into fuzzy events. These incorporate Wald's notion of subjective qualifiers. When subjective probability distributions are conjoined by OR and AND operations, it is efficacious to apply Zadeh's extension principle to integral transforms of these subjective qualifiers. A future task is to establish a fuzzy Bayes' theorem for cases in which OR- and AND-conjunctions are simultaneously present.

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