Asymmetric Cournot Duopoly: A Game Complete Analysis

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Abstract: In this paper, we apply the *Complete Analysis of Differentiable Games* (introduced by D. Carfì in Topics in Game Theory (2012), Carfì ICT 2009, Carfì AAPP 2009, Carfì GO 2009; already employed by himself and others in Carfì TPREF 2011, Carfì AAPP 2010, Carfì ISGC 2009) and some new algorithms, using the software *wxMaxima 11.04.0*, in order to reach a *total scenario knowledge (that is the total knowledge of the payoff space of the interaction)* of the classic Cournot Duopoly (1838), viewed as a complex interaction between two competitive subjects, in a particularly interesting *asymmetric case*. Moreover, in this work we propose a theoretical justification, for a general kind of asymmetric duopolistic interactions (which often appear in the real economic world), by considering and proposing a *Cobb-Douglas perturbation* of the classic linear model of production costs.

Keywords: Asymmetric Cournot Duopoly, Software algorithms in Microeconomic Policy, Complete Analysis of a normal-form game, valuation of Nash equilibriums, Bargaining solutions.

1. INTRODUCTION

The Cournot Duopoly is a classic oligopolistic market in which there are two enterprises producing the same commodity and selling it in the same market. In this classic model, in a competitive background, the two enterprises employ as possible strategies the quantities of the commodity produced. The main solutions proposed in economic and Game Theory literature (see for instance Aubin 1982, Aubin 1998, Dennis *et al.*) for this kind of duopoly are the *Nash equilibrium* (see also Carfì *et al.* AAPP 2009, Carfì SIMAI 2008, Carfì MPRA 29001, Carfì MPRA 28971) and the *Collusive Optimum*, without any subsequent critical exam about these two kinds of solutions (see, on the contrary, Carfì TPREF 2011, Carfì AAPP 2010, Carfì ICT 2009, Carfì ISGC 2009, Carfì AAPP 2009, Carfì GO 2009. The absence of any critical quantitative analysis is due to the relevant lack of knowledge regarding the set of all possible outcomes of this strategic interaction and in particular of the Pareto boundary of the problem (see also Carfì AAPP 2008). On the contrary, by considering the Cournot Duopoly as a *differentiable game* (normal form games with differentiable payoff functions) and studying it by the topological methodologies introduced in Game Theory by D. Carfì, we obtain an exhaustive and complete vision of the entire payoff space of the Cournot game (this also in asymmetric cases with the help of wxMaxima) and this total view allows us to analyze critically the classic solutions and to find other ways of action to select Pareto strategies, in the *asymmetric cases too*. In order to illustrate the applications of this topological methodologies to the considered infinite game, several compromise decisions are considered, and we show how the complete study gives a real extremely extended comprehension of the classic model.

2. FORMAL DESCRIPTION OF A GENERALIZED COURNOT NORMAL FORM GAME

Our model of Cournot game is a non-linear two-players loss game G of type (f, >) (see also Carfì ICT 2009, Carfì AAPP 2009 and Carfì GO 2009). The two players/enterprises are called *Emil* and *Frances* (following J.P. Aubin's books 1982 and Aubin 1998).

Assumption 1 (Strategy Sets). The two players produce and offer the same commodity in the quantities $x \in \mathbb{R}_{\geq}$ for Emil and $y \in \mathbb{R}_{\geq}$ for Frances. In more precise terms: the payoff function f of the game G is defined on a subset of the positive cone of the Cartesian plane $\mathbb{R}^2$, interpreted as a space of bi-quantities. We assume (by simplicity) that the set of all strategies, of each player, is the interval E = [0, +∞].

Assumption 2 (Asymmetry of the Game). The game G is not assumed necessarily symmetric with respect to the players. In other terms, the payoff pair $f(x, y)$ is not assumed to be the symmetric of the pair $f(y, x)$.

Assumption 3 (Form of Price Function). We suppose the price function, p from $\mathbb{R}^2$ into $\mathbb{R}$, linear and defined by

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$$p(x, y) = a - b_1x - b_2 y \qquad (2.1)$$

for every productive bi-strategy (x, y), where $a \geq 0$ is a fixed price and $b_i > 0$ ($i = 1,2$) is the marginal coefficient corresponding to the production of the player i.

Assumption 4 (Form of Cost Functions). Let the cost function C_1 (defined on E^2) of the first player be given by

$$C_1(x, y) = c_1 x + e_1xy + d, \qquad (2.2)$$

for every positive price pair (x, y) and let, analogously, the demand function of the second enterprise be given by

$$C_2(x, y) = c_2 y + e_2 xy + d, \qquad (2.3)$$

for every positive bi-strategy (x, y), where $c = (c_1, c_2) > 0$ is the pair of marginal costs and where the number $d \geq 0$ is the fixed cost.

Modeling Axiom. So, we consider a *Cobb-Douglas perturbation* of the classic linear costs.

This kind of modeling should be explained from an economic point of view.

The economic interpretation is the following:

- our model represents those economic duopolies in which the production of any player may also influence the production cost of the other player;
- if e_i *is negative,* our model represents those economic duopolies in which a combined production of the same good, by the two actors, determines a positive effect upon the costs (that is, it determines a decrease in costs, thanks to virtuous effects on the improvement of technologies or in obtaining raw materials), especially for positive effects on research and development of new methods of production. Effects that, often (or almost always), are evident in the most of economic sectors.

Assumption 5 (Payoff Functions). Setting

$w_i := a - c_i$,

the first player's *net cost function* is defined, classically, by the revenue

$$f_1(x, y) = C_1(x, y) - p(x, y)x =$$
$$= c_1 x + e_1xy + d - (ax - b1x^2 - b_2 xy) =$$
$$= x(b_1x + (b_2 + e_1)y - (a - c_1)) + d = \qquad (2.4)$$
$$= x(b_1x + (b_2 + e_1)y - w_1)+d =$$
$$= w_1 x ((b_1/w_1)x + ((b_2 + e_1)/w_1)y -1) + d,$$

for every positive bi-strategy (x, y).

Symmetrically, for Frances, the net cost function is defined by

$$f_2(x, y) = C_2(x, y) - p(x, y)y =$$
$$= y((b_1 + e_2)x + b_2y - (a - c_2)) + d = \qquad (2.5)$$
$$= y((b_1 + e_2)x + b_2y - w_2) + d =$$
$$= w_2y((b_1 + e_2)/w_2)x + (b_2/w_2)y - 1) + d,$$

for every positive bi-strategy (x, y).

3. STUDY OF THE COURNOT'S NORMAL FORM GAME

In the following we shall study the following particular case. We shall put:

$w_2 = w_1 = 1$; $b_1 = 2$; $e_2 = - 1$; $b_2 = 1$; $e_1 = 0$,

so that, the bi-loss function is defined by

$$f(x, y) = (x(2x + y - 1), y(x + y - 1)) + (d, d),$$

for every bi-strategy (x, y) of the game in the unbounded square E^2.

Remark. Similar examples, of asymmetric duopolies, are already considered by Vannoni and Piacenza, in some practical applications and - in general - in Dennis; the authors do not justify the form of the payoff functions. We justify the asymmetric form by using our Cobb-Douglas perturbation of the classic cost function, which implies a decrease in costs, given by the positive effect of new technologies (or methodologies) for the production of the good and for the use and finding of the raw materials for the productions.

Payoff Vector-Function. When the fixed cost d is zero (this assumption determines only a "reversible" translation of the loss space), the bi-loss function f from the compact square $[0,1]^2$ into the bi-loss plane $\mathbb{R}^2$ is defined by

$$f(x, y) = (x(2x + y - 1), y(x + y - 1)), \qquad (3.1)$$

for every bi-strategy (x, y) in the square S = $[0, 1]^2$, which is the convex envelope of its vertices, denoted by A, B, C, D starting from the origin and going anticlockwise.

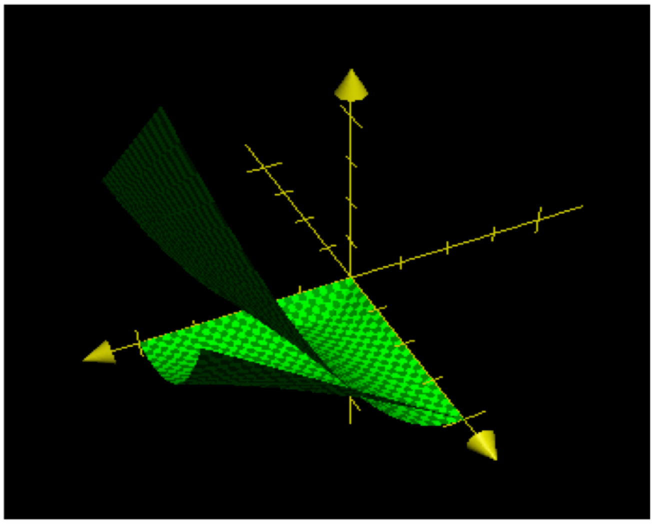

Figure 1: The graphical representation of the Cournot payoff function f in the space R^3, with respect to an orthonormal basis.

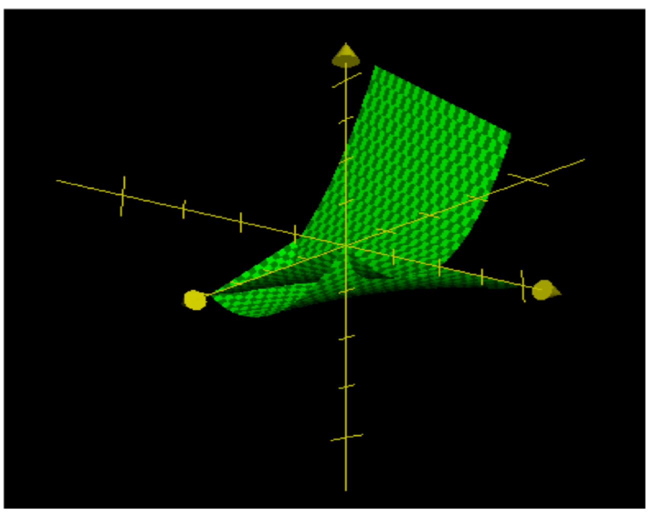

Figure 2: The graphical representation of the Cournot payoff function f in the space R^3, with respect to an orthonormal basis.

Critical Space of the Game. Now, we must find *the critical space of the game* and its image by the function f, before representing f(S). We determine (as explained in Carfì's *Topics in Game Theory* (2012), Carfì ICT 2009, Carfì AAPP 2009 and Carfì GO 2009) firstly the *Jacobian matrix* of the function f at any point (x, y) $\in$ S - denoted by $J_f(x, y)$. We will have, in vector form, the pair of gradients

$$J_f(x, y) = ((y+4x-1, x), (y, 2y+x-1)), \qquad (3.\ 2)$$

and concerning the determinant of the above pair of vectors

$$\det J_f(x, y) = (y+4x-1)(2y+x-1)-xy =$$

$$= 2y^2 + 8xy - 3y + 4x^2 - 5x + 1. \qquad (3.\ 3)$$

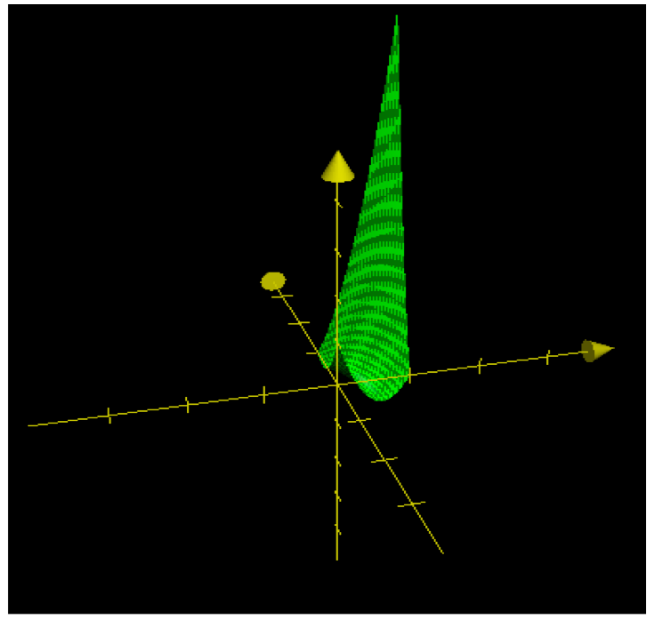

Figure 3: The graphical representation of the Jacobian determinant of payoff function *f.*

The *Jacobian determinant* is zero at those points (x_1, y_1) and (x_2, y_2) of the strategy square such that

$$y_1 = -(1/4)\ (32x_1^2-8x_1+1)^{1/2} - 2x_1+{}^3/_4 \qquad (3.\ 4)$$

or

$$y_2 = (1/4)\ (32x_2^2-8x_2+1)^{1/2} - 2x_2+ {}^3/_4. \qquad (3.\ 5)$$

We obtain two curves (Figure **4**) whose union is *the critical zone of Cournot Game*.

4. TRANSFORMATION OF THE STRATEGY SPACE

It is readily seen that the intersection points of the green curve with the boundary of the strategic square are the point K = $({}^2/_8, 0)$.

Remark

The point K is the intersection point of the *green curve* with the *segment* [A, B], its abscissa μ verifies the non-negative condition and the following equation

$$(32\mu_1^2 - 8\mu_1 + 2)^{1/2} = 3 - 8\mu_1, \qquad (4.\ 1)$$

this abscissa is so $\mu_1 = {}^2/_8$.

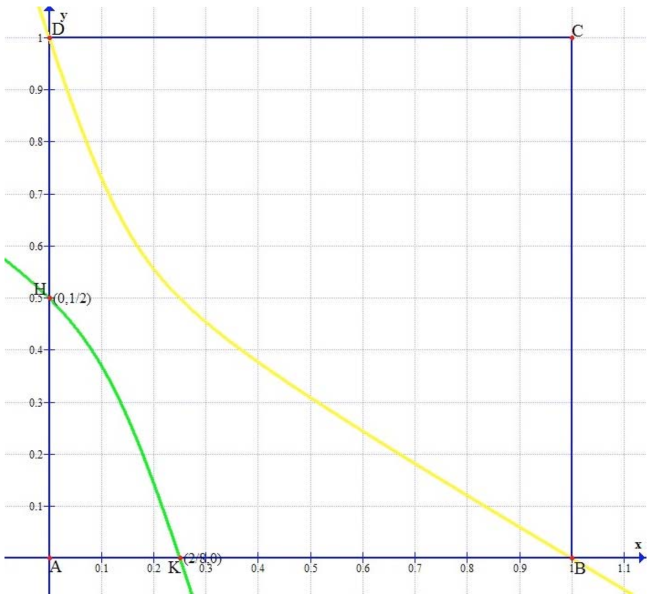

Figure 4: Critical zone of Cournot game.

The point H is the intersection point between the *ordinate axis* and the *critical part*

$y_1 = -(1/4)(32x_1^2 - 8x_1+1)^{1/2} - 2x_1 + {}^3/_4$.

We start from Figure **4**. The transformations of the bi-strategy square vertices and of the points H, K are the following:

A' = f(A) = f(0, 0) = (0, 0);

B' = f(B) = f(1, 0) = (1, 0);

C' = f(C) = f(1, 1) = (2, 1);

D' = f(D) = f(0, 1) = (0, 0);

H' = f(H) = $f(0, {}^1/_2) = (0, -{}^1/_4)$;

K' = f(K) = $f({}^2/_8, 0) = (-{}^1/_8, 0)$.

Starting from Figure **4**, with S = $[0, 1]^2$, we can do the transformation of its sides.

Side [A, B]. Its parameterization is the function sending any point $x \in [0, 1]$ into the point (x, 0); the transformation of this side can be obtained by transformation of its generic point (x, 0), we have

$$f(x, 0) = (2x^2 - x, 0). \quad (4.2)$$

We obtain the segment with end points K' and B', with parametric equations

$$X = 2x^2 - x \text{ and } Y = 0, \quad (4.3)$$

with x in the unit interval.

Side [B, C]. Its parameterization is

$(x = 1, y \in [0, 1])$;

the figure of the generic point is

$$f(1, y) = (y + 1, y^2). \quad (4.4)$$

We can obtain the parabola passing through the points B', C' with parametric equations

$$X = y + 1 \text{ and } Y = y^2. \quad (4.5)$$

Side [C, D]. Its parameterization is

$(x \in [0, 1], y = 1)$;

the transformation of its generic point is

$$f(x, 1) = (2x^2, x). \quad (4.6)$$

We obtain the parabola passing through the points D', C' with parametric equations

$$X = 2x^2 \text{ and } Y = x, \quad (4.7)$$

with x in the unit interval.

Side [A, D]. Its parameterization is

$(x = 0, y \in [0, 1])$;

the transformation of its generic point is

$$f(0, y) = (0, y^2 - y). \quad (4.8)$$

We obtain the segment with end points A' and H', with parametric equations

$$X = 0 \text{ and } Y = y^2 - y. \quad (4.9)$$

Now, we find *the transformation of the critical zone*. The parameterization of the critical zone is defined by the equations

$y_1(x) = -(1/4)(32x^2 - 8x+1)^{1/2} - 2x + {}^3/_4$,[1]

and

$y_2(x) = (1/4)(32x^2-8x+1)^{1/2}-2x + {}^3/_4$.[2]

The parametrization of the *green zone* is

$(x \in [0, 1], y = y_1(x))$;

[1] Equation 3.4 pag. 3.
[2] Equation 3.5 pag. 3.

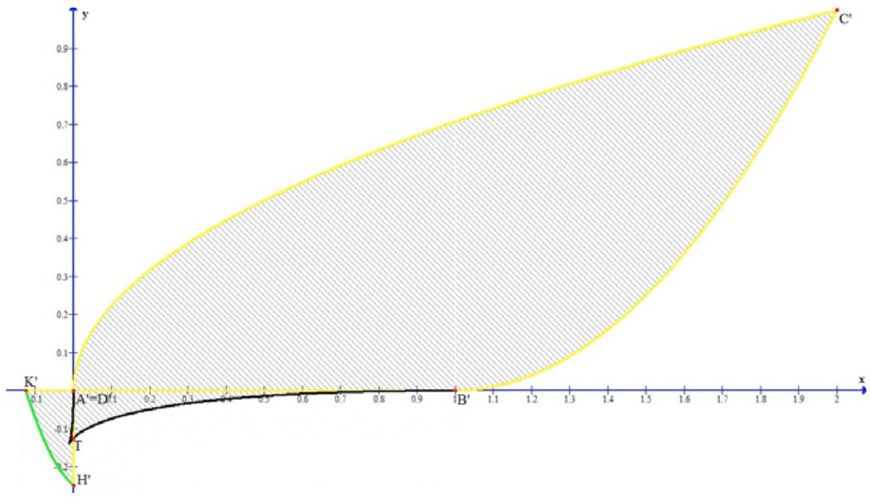

Figure 5: Payoff space of Cournot game.

the transformation of its generic point is

$$f(x, y_1) = (x(2x+y_1-1), y_1(x+y_1-1)), \quad (4.10)$$

a parametrization of the *yellow zone* is ($x \in [0, 1]$, $y = y_2$); the transformation of its generic point is

$$f(x, y_2) = (x(2x+y_2-1), y_2(x+y_2-1)). \quad (4.11)$$

The transformation of the green zone is determined by formulas

$$X = x\,(-(1/4)\,(32x^2 - 8x+1)^{1/2} - {}^1/_4) \quad (4.12)$$

and

$$Y = (-0.25(32x^2 - 8x + 1)^{1/2} - 2x + {}^3/_4)(-\,0.25\,(32x^2 - 8x + 1)^{1/2} - x - {}^1/_4), \quad (4.13)$$

and the transformation of the yellow zone is determined by formulas

$$X = x((1/4)\,(32x^2-8x+1)^{1/2} - {}^1/_4) \quad (4.14)$$

and

$$Y = (0.25(32x^2 - 8x + 1)^{1/2} - 2x + {}^3/_4)(0.25(32x^2 - 8x + 1)^{1/2} - x - {}^1/_4). \quad (4.15)$$

We have two colored curves in *green* and *black* (Figure **5**). The *black* curve is break by a point of discontinuity T obtained by resolving the following equation

$$x\,((1/4)\,(32x^2 - 8x + 1)^{1/2} - {}^1/_4) = 0; \quad (4.16)$$

the solutions of the above equation are

$$x_1 = {}^1/_4,\ x_2 = 0, \quad (4.17)$$

and, replacing them in the parametrical equations of the critical zone (4.15) we obtain $T_1 = 0$ and $T_2 = -\,{}^1/_8$

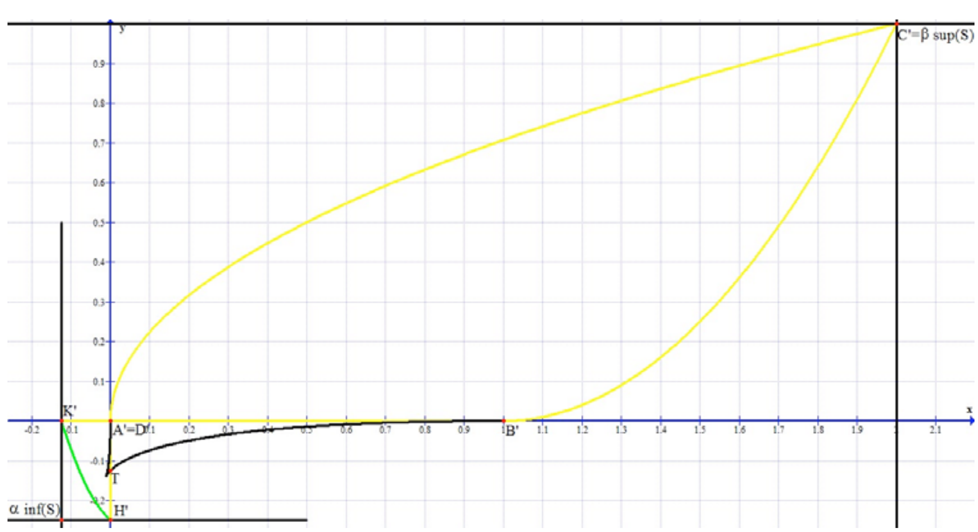

Figure 6: Extrema of the Asymmetric Cournot game.

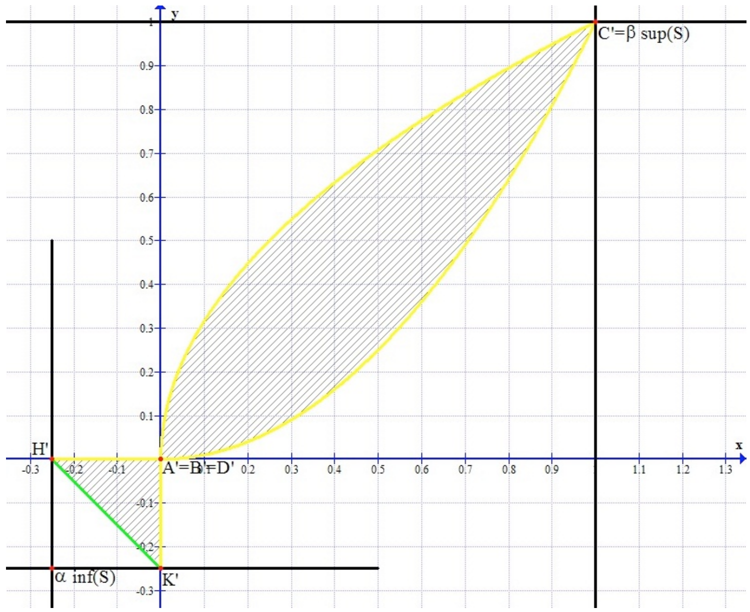

Figure 7: Extrema of the Symmetric Cournot game.

5. NON-COOPERATIVE FRIENDLY PHASE

Examining the Figure **5** we note that the game has two extremes: the *shadow minimum* $\alpha = (-^1/_8, -^1/_4)$ and the *maximum* β = C'= (2, 1). The *Pareto minimal boundary of the payoff space* f(S) is the curve passing through the points K' and H' colored in *green* showed in the Figure **6**. The *Pareto maximal boundary of the payoff space* f(S) coincides with the *maximum* β = C'= (2, 1). Both Emil and Frances do not control the Pareto minimal boundary; they could reach the point K' and H' of the boundary, but the solution is not satisfactory for them. In fact, a player will suffer the maximum loss.

Remark. Comparing the Figure **6** with the Figure **7** we may observe that the benefit to the community decreases in case of asymmetry; in fact the area contained in the first quadrant is greater than in the symmetric case and the area contained in the third quadrant is smaller than in the symmetric case. In other terms, when the Cournot duopoly becomes an asymmetric games is easier to have a loss.

6. PROPERLY NON-COOPERATIVE (EGOISTIC) PHASE

Now, we will consider *the best reply correspondences* between the two players Emil and Frances. If Frances produces the quantity y of the commodity, Emil, in order to reply rationally, should minimize his *partial cost function*

$$f_1(\cdot, y) : x \mapsto x(2x + y - 1), \tag{6. 1}$$

on the compact interval [0,1]. According to the Weierstrass theorem, there is at least one Emil's strategy minimizing that partial cost function and, by Fermàt theorem, the *Emil's best reply strategy to Frances' strategy y* is the only quantity

$$B_1(y) = x^* := (1/4)(1 - y). \tag{6. 2}$$

Indeed, the partial derivative

$$f_1(\cdot, y)'(x) = 4x + y - 1, \tag{6. 3}$$

is negative for $x < x^*$ and positive for $x > x^*$. So, the Emil's best reply correspondence is the function B_1 from the interval [0, 1] into the interval [0, 1], defined by $y \mapsto {}^1/_4(1 - y)$. If Emil produces the quantity x of the commodity, Frances, in order to reply rationally, should minimize his *partial cost function*

$$f_2(x, \cdot) : y \mapsto y(x + y - 1), \tag{6. 4}$$

on the compact interval [0, 1]. According to the Weierstrass theorem, there is at least one Frances' strategy minimizing that partial cost function and, by Fermàt theorem, the *Frances' best reply strategy to Emil's strategy x* is the only quantity

$$B_2(x) = y^* := {}^1/_2(1 - x). \tag{6. 5}$$

Indeed, the partial derivative

$$f_2(x, \cdot)'(y) = 2y + x - 1, \tag{6. 6}$$

is negative for $x < x^*$ and positive for $x > x^*$. So, the Frances' best reply correspondence is the function B_2

from the interval [0, 1] into the interval [0, 1], defined by $x \mapsto {}^1/_2(1 - x)$. The *Nash equilibrium* is the fixed point of the multifunction B - associated with the pair of two reaction functions (B_2, B_1) - defined from the Cartesian product of the domains into the Cartesian product of the codomains (in inverse order), by B : $(x, y) \mapsto (B_1(y), B_2(x))$, that is the only bi-strategy (x, y) such that

$$x = ({}^1/_4)(1 - y) \text{ and } y = {}^1/_2(1 - x), \tag{6. 7}$$

that is the point N = $({}^1/_7, {}^3/_7)$ - as we can see also from Figure **8** - which gives a bi-loss N' = $(-{}^2/_{49}, -{}^9/_{49})$, as Figure **9** will show. The Nash equilibrium is not completely satisfactory, because it is not a Pareto optimal bi-strategy, but it represents the only properly non-cooperative game solution.

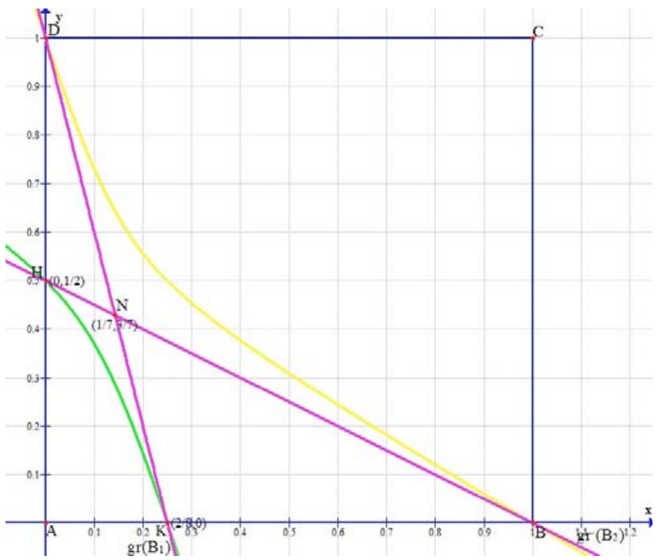

Figure 8: Nash Equilibrium of Cournot game.

7. DEFENSIVE AND OFFENSIVE PHASE

Players' conservative values are obtained through their *worst loss functions*.

Worst loss functions. On the square S = $[0, 1]^2$, *Emil's worst loss function* is defined by

$$f^{\#}_1(x) = \sup_{y \in F} x\,(2x + y - 1) = 2x^2, \tag{7. 1}$$

its minimum will be

$$v^{\#}_1 = \inf_{x \in E} (f^{\#}_1(x)) = \inf_{x \in E} 2x^2 = 0, \tag{7. 2}$$

attained at the conservative strategy $x^{\#} = 0$.

Frances' worst loss function is defined by

$f^{\#}_2(y) = \sup_{x \in E} y\,(x + y - 1) = y^2$, (7. 3)

its minimum will be

$$v^{\#}_2 = \inf_{y \in F} f^{\#}_2(y) = \inf_{y \in F} y^2 = 0 \tag{7. 4}$$

attained at the unique conservative strategy $y^{\#} = 0$.

Conservative bivalue. The *conservative bivalue* is

$v^{\#} = (v^{\#}_1, v^{\#}_2) = (0, 0)$.

The worst offensive multi-functions are determined by the study of the *worst loss functions*.

The Frances' worst offensive reaction multifunction O_2 is defined by $O_2(x) = 1$, for every Emil's strategy x; indeed, fixed an Emil's strategy x the Frances' strategy 1 maximizes the partial cost function $f_1(x, .)$. The Emil's

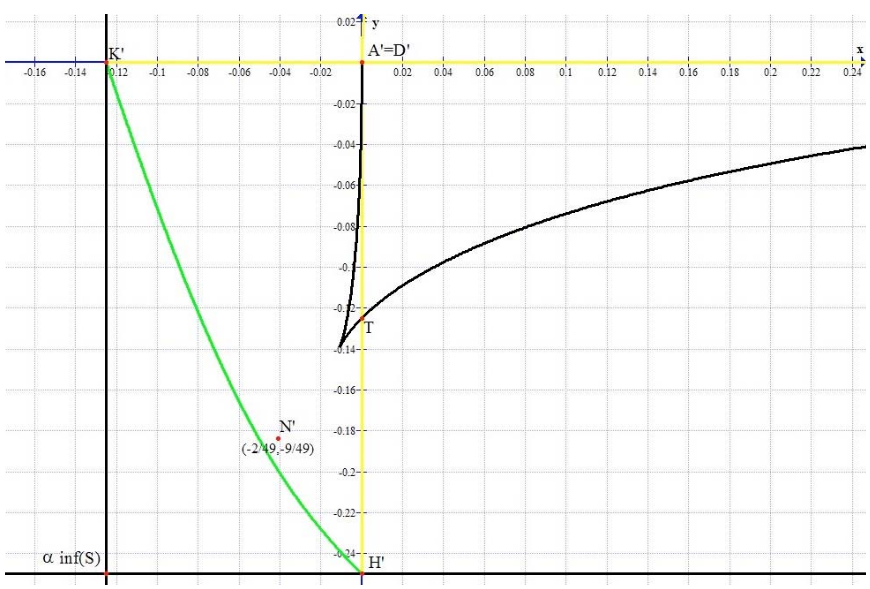

Figure 9: Payoff at Nash equilibrium of Cournot game.

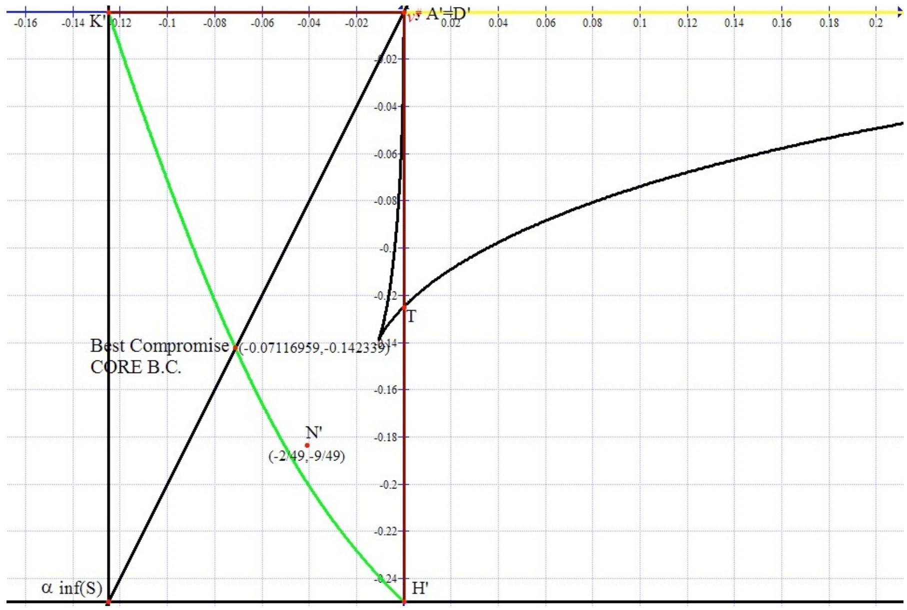

Figure 10: Conservative study. The core and the Kalai-Smorodinsky payoff of Cournot game.

worst offensive correspondence versus Frances is defined by $O_1(y) = 1$, for every Frances' strategy y.

The dominant offensive strategy is 1 for both players, indeed the offensive correspondences are constant.

The core of the payoff space (in the sense introduced by J.P. Aubin) is the Pareto minimal boundary, contained in the cone of lower bounds of the conservative bi-value $v^{\#}$; the conservative bi-value don't give us a bound for the choice of cooperative bi-strategies.

The defensive knot of the game is the point (0,0).

8. COOPERATIVE PHASE

When there is an agreement between the two players, *the best compromise solution (in the sense introduced by J.P. Aubin)* showed graphically in the Figure **10**.

Besides, the best compromise solution coincides with the *core best compromise*, with the *Nash bargaining solution*, with the *Kalai-Smorodinsky bargaining solution*. It coincides also with *the transferable utility solution* which is the unique Pareto strategy that minimized the aggregate utility function $f_1 + f_2$, this can be easily viewed by geometric evidences considering on the payoff universe the levels of that aggregate function, which are affine lines parallel to the vector (1, -1).

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