Instrumental Variables Estimation of Systems of Simultaneous Equations: Interrelation of Methods

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Abstract: The article is devoted to the interrelation between methods of estimating parameters of simultaneous equations. Simultaneous equations model (SEM) is commonly used to model complex socio-economic phenomena. SEM is a set of linear simultaneous equations in which response variables are among explanatory variables in each equation of regression. This causes the problem of endogeneity and leads to biased and inconsistent estimation of parameters. There is a number of special methods to solve the problem of endogeneity of regressors: method of instrumental variables (IV), indirect least squares method (ILS), two-stage least squares method (2SLS), and three-stage least squares method (3SLS). In this article, the relationship between 2SLS and IV, ILS and 2SLS, ILS and OLS with restrictions on structural parameters, as well as the equivalence of point estimates of parameters and autocovariance matrices, is shown using empirical example.

Keywords: System of simultaneous equations, instrumental variables method, parameter constraints, parameter estimates, autocovariance matrix of parameter estimates.

INTRODUCTION

Systems of Equations: Specification and Estimation Methods

In Econometrics, different systems of equations are used to describe many complex economic problems:

- Systems of independent equations (variables in individual equations are independent).
- Systems of seemingly unrelated equations, SUR (disturbances are contemporaneously correlated across equations).
- Systems of simultaneous equations, SEM (one variable which is endogenous for one equation can be regressor for the other) [Greene 2011], [Eliseeva, Kurysheva, Kosteeva 2008].

In economic systems, in general, changes in some variables cannot occur with the absolute immutability of others, therefore, when formalizing them, the SEM is used more often than other systems. Two forms of specification of models of SEM are applied:

1. Structural form (the result of formalization)

$$A \cdot Y_t + B \cdot X_t = V_t \quad , \tag{1}$$

where A, B — matrices of structural parameters,

$Y_t = \left(Y_{1t}, Y_{2t}, ..., Y_{mt}\right)^T$ — column vector of the values of endogenous variables[1], $X_t = \left(X_{1t}, X_{2t}, ..., X_{kt}\right)^T$ — column vector of the values of predetermined variables, which might include exogenous variables and lagged values of endogenous variables as well. $V_t = \left(v_{1t}, v_{2t}, ..., v_{mt}\right)^T$ — column vector of the random disturbances,

t — observation's number;

2. Reduced form (when the vector of endogenous variables is expressed as an explicit function of a predetermined variables):

$$Y_t = -A^{-1}BX_t + A^{-1}V_t = MX_t + U_t \; , \tag{2}$$

where $M = -A^{-1}B$ — matrix of coefficients of reduced form, (3)

and $U_t = A^{-1}V_t$ — vector of the random disturbances of reduced form. (4)

Two special econometric methods were developed to estimate parameters of the system of simultaneous equations: indirect least square method (ILS) and two stage least square method (2SLS). They allow to solve the problem of endogeneity of variables that appears during the estimation of the structural parameters of the system. To increase the precision of 2SLS estimators of structural parameters, we use the three stage least square method (3SLS) that considers a correlation of disturbances of some of the equations in the system [Kleiber, Zeileis 2008].

Let's compare different estimates of parameters of simultaneous equation model and their autocovariance matrices obtained by using all the methods listed above.

Consider the methods for estimating the parameters of the SEM on the example of the cheese market data in Russia:

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[1]The symbol «*T*» is used for transposition of matrices.

Table 1: Data

t (year)	Y_t	P_t	P_{t-1}	X_t
2001	390	87,908	85,170	5325,8
2002	471	102,626	103,060	6831
2003	558	110,192	102,670	8900,5
2004	625	122,684	111,950	10976,3
2005	689	138,597	122,300	13819
2006	720	159,944	138,720	17290,1
2007	758	177,839	144,260	21311,5
2008	746	238,017	233,930	25231,6
2009	770	239,444	212,920	28452,3
2010	808	254,717	213,110	32498
2011	820	291,938	263,200	35649
2012	858	313,087	273,430	39904
2013	849	330,454	272,570	44650
2014	789	370,271	326,890	47921
2015	762	422,698	388,810	53526
2016	639	440,027	418,610	54117
2017	641	466,577	461,710	55368

$$\begin{cases} Y_t = a_0 + a_1 \cdot P_t + a_2 X_t + \varepsilon_{1t}, \\ Y_t = b_0 + b_1 \cdot P_t + b_2 P_{t-1} + \varepsilon_{2t}. \end{cases} \quad (5)$$

The vector of endogenous variables (5) includes the following components:

Y_t – volume of transactions (thousand tons); P_t – average consumer price for cheese (rubles per kilogram);

The vector of predefined (exogenous and lagged) variables: X_t – money income of population (ruble, billions); P_{t-1} – lagged variable of average consumer price for cheese (rubles per kilogram).

Data[2] for the period 2001 to 2017 are given in Table **1**:

Method of Instrumental Variables (IV)

The method of instrumental variables can be used to solve the endogeneity problem in SEM, which leads to biased and inconsistent estimation. To replace regressors correlated with disturbances, instrumental variables that are strongly correlated with a regressors, but not correlated with the model's disturbances, are used. Generalized instrumental variables estimator (GIVE) [Abbes 2007]:

$$\hat{\beta}_{IV} = \left(X^T P_Z X\right)^{-1} X^T P_Z Y \ , \quad (6)$$

where $P_Z = Z\left(Z^T Z\right)^{-1} Z^T$ — projection to subspace of instrumental variables,

Z – matrix of instrumental variables, which is used when the number of instrumental variables is greater than the number of replaced regressors ($p > k$). Autocovariance matrix of parameter estimates is calculated according to the formula [Magnus, Katyshev, Peresetzkyi, Golovan 2007]

$$\hat{C}_{\hat{\beta}\hat{\beta}} = \hat{\sigma}^2 \left(X^T P_Z X\right)^{-1} = \hat{\sigma}^2 \left(X^T Z \left(Z^T Z\right)^{-1} Z^T X\right)^{-1} . \quad (7)$$

If $p = k$, then matrix $X^T Z$ is a square nonsingular matrix, and the least-square estimate (6) will be transformed into IV-estimate

$$\hat{\beta}_{IV} = \left(X^T Z \left(Z^T Z\right)^{-1} Z^T X\right)^{-1} X^T Z \left(Z^T Z\right)^{-1} Z^T Y =$$

$$\left(Z^T X\right)^{-1} \left(Z^T Z\right) \left(X^T Z\right)^{-1} X^T Z \left(Z^T Z\right)^{-1} Z^T Y = \left(X^T Z\right)^{-1} X^T Y \ . \quad (8)$$

The problem of choosing instrumental variables is not always easily solved. One way is to replace a variable with its estimate. In relation to the first equation of the structural form of the SEM (5), we replace the variable P_t with its estimate $\hat{P}_t$ obtained from the regression model based on all predefined variables of the model

[2] Russian statistical yearbook. 2018. http://www.gks.ru/bgd/regl/b18_13/Main.htm

Table 2: Matrix IV-Estimates

Matrix of regressors *X*			Matrix of instrumentalvariables *Z*		
1	103,06	5325,8	1	87,908	5325,8
1	102,67	6831,0	1	102,626	6831,0
1	111,95	8900,5	1	110,192	8900,5
1	122,30	10976,3	1	122,684	10976,3
1	138,72	13819,0	1	138,597	13819,0
1	144,26	17290,1	1	159,944	17290,1
1	233,93	21311,5	1	177,839	21311,5
1	212,92	25231,6	1	238,017	25231,6
1	213,11	28452,3	1	239,444	28452,3
1	263,20	32498,0	1	254,717	32498,0
1	273,43	35649,0	1	291,938	35649,0
1	272,57	39904,0	1	313,087	39904,0
1	326,89	44650,0	1	330,454	44650,0
1	388,81	47921,0	1	370,271	47921,0
1	418,61	53526,0	1	422,698	53526,0
1	461,71	54117,0	1	440,027	54117,0
1	478,88	55368,0	1	466,577	55368,0

$$\hat{P}_t = \hat{m}_{31} + \hat{m}_{32}X_t + \hat{m}_{33}P_{t-1} = \underset{(13,672)}{24,738} + \underset{(0,001)}{0,004} \cdot X_t + \underset{(0,207)}{0,507} \cdot P_{t-1} \quad (9)$$

where X_t – money income of population, P_{t-1} – lagged variable of average consumer price for cheese. The elements of the matrices of regressors and instrumental variables of the first equation of the SEM (5) are given in the Table **2**:

The result of estimation of SEM using instrumental variables (6)-(8):

$$\hat{Y}_t = \underset{(69,420)}{798,021} - \underset{(1,359)}{5,321} \cdot P_t + \underset{(0,010)}{0,042} \cdot X_t\,, \quad R^2 = 0,680\,, \quad (10)$$

$$\hat{C}_{\hat{\beta}\hat{\beta}} = \hat{\sigma}^2 \left(X^T Z \left(Z^T Z \right)^{-1} Z^T X \right)^{-1} =$$

$$= \begin{pmatrix} 4819,186 & -78,378 & 0,516 \\ -78,378 & 1,847 & -0,013 \\ 0,516 & -0,013 & 9,350\text{e-}05 \end{pmatrix}. \quad (11)$$

In the theory of simultaneous equations, this procedure was called two-step least square method (2SLS). Estimates of the endogenous regressors of the equation based on all predetermined variables of the model made on the first step, will be used to estimate structural variables on the second step.

Indirect Least Squares Method (ILS)

ILS is used for exactly identified equations and consists of the following steps:

1. Build the reduced form (2)-(4) using the structural form (1);

2. Determine OLS-estimates of parameters of the reduced form using the Table **1** data:

$$\left.\begin{aligned} \hat{Y}_t &= \hat{m}_{11} + \hat{m}_{12}X_t + \hat{m}_{13}P_{t-1} = \underset{(45,515)}{666,389} + \underset{(0,005)}{0,022} \cdot X_1 - \underset{(3,274)}{2,698} \cdot X_2, \quad R^2 = 0,680 \\ \hat{P}_t &= \hat{m}_{31} + \hat{m}_{32}X_t + \hat{m}_{33}P_{t-1} = \underset{(13,672)}{24,738} + \underset{(0,001)}{0,004} \cdot X_1 + \underset{(0,207)}{0,507} \cdot X_2 \quad R^2 = 0,969 \end{aligned}\right\} \quad (12)$$

3. Calculate the estimates of structural parameters using OLS – estimates (12) and equation of relation between structural and reduced parameters (3).

For convenience, we write equation (3) using the extended matrix of structural form coefficients $\bar{A} = \left(A \middle| B \right)$ [Babeshko, Bich, Orlova 2017]:

$$\bar{A} \cdot \begin{pmatrix} M \\ I \end{pmatrix} = 0\,, \quad (13)$$

where *I* — the identity matrix $k \times k$, and *k* is the number of columns of the matrix of reduced form coefficients. If the values of the elements of the matrix *M* of the reduced form are unknown, then their LSM-estimates in the system (13) are used. The expanded matrix of structural parameters for model (5) has the form:

$$\overline{A} = (A|B) = \left(\begin{array}{ccc|ccc} 1 & 0 & -a_1 & -a_0 & -a_2 & 0 \\ 0 & 1 & -b_1 & -b_0 & 0 & -b_2 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{array} \right). \qquad (14)$$

First and second equations of the system (5) are exactly identified, therefore ILS is applicable. Let's take a closer look at the first equation of the structural form. Considering (14), its parameters satisfy the system of linear equations (13)

$$\begin{pmatrix} 1 & 0 & -a_1 & -a_0 & -a_2 & 0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

or

$$\begin{array}{c} m_{11} - a_1 m_{31} - a_0 = 0 \\ m_{12} - a_1 m_{32} - a_2 = 0 \\ m_{13} - a_1 m_{33} = 0 \end{array}. \qquad (15)$$

Thus, ILS – estimates of the structural parameters of the first equation of the system (5) are equal to

$$\left. \begin{array}{c} \hat{a}_1 = \frac{\hat{m}_{13}}{\hat{m}_{33}} = \frac{-2,698}{0,507} = -5,321, \\ \hat{a}_2 = \hat{m}_{12} - \hat{a}_1 \hat{m}_{32} = 0,022 + 5,321 \cdot 0,004 = 0,042, \\ \hat{a}_0 = \hat{m}_{11} - \hat{a}_1 \hat{m}_{31} = 666,389 + 5,321 \cdot 24,738 = 798,020 \end{array} \right\} (16)$$

and coincide with their 2SLS – estimates (10).

Estimation of SEM with Parameter Restrictions

To determine the autocovariance matrix of the ILS-estimates of the parameters, we proceed as follow. Let us show the connection between ILS and OLS with the restrictions on structural parameters. OLS – estimate of the parameters of the linear model of multiple regression

$$Y = X\beta + \varepsilon \qquad (17)$$

with the general linear constraints

$$H_0 : H\beta = r, \qquad (18)$$

where Y — vector of the values of endogenous variables $(n \times 1)$, X — matrix of regressors $(n \times k)$, β — vector of parameters $(k \times 1)$, r — vector of constants of constraints $(q \times 1)$ [3], H — constraints matrix $(q \times k)$, $ranr(H) = q$, $q \le k$, determined by [Magnus, Katyshev, Peresetzkyi, Golovan 2007]

$$\hat{\beta}_R = \hat{\beta}_{UR} + b, \qquad (19)$$

where $\hat{\beta}_R$ — restricted OLS-estimate; $\hat{\beta}_{UR}$ — unrestricted OLS-estimate

$$\hat{\beta}_{UR} = \left(X^T X\right)^{-1} X^T Y, \qquad (20)$$

b — correction factor

$$b = (X^T X)^{-1} H^T (H(X^T X)^{-1} H^T)^{-1} (r - H\hat{\beta}_{UR}) =$$

$$= (X^T X)^{-1} H^T V (r - H\hat{\beta}_{UR}) = Kr - KH\hat{\beta}_{UR}, \qquad (21)$$

where

$$V = \left[H(X^T X)^{-1} H^T \right]^{-1}, \ K = (X^T X)^{-1} H^T V. \qquad (22)$$

The estimate $\hat{\beta}_R$ (18) of the parameters of the model (17), is the solution to the conditional extremum problem:

$$e^T e = (Y - X\hat{\beta}_R)^T (Y - X\hat{\beta}_R) \to \min, \ H\hat{\beta}_R = r,$$

where *e* — vector of the residuals of the model with the restrictions on parameters.

Let us estimate the first equation of the model (5) using OLS without restrictions:

$$Y_t = \underset{(51,785)}{691,491} - \underset{(0,772)}{2,811} \cdot P_t + \underset{(0,006)}{0,024} \cdot X_t + \underset{(82,57)}{e_t}, \quad R^2 = 0,656.$$

OLS-estimates of the structural parameters are:

$$\hat{\beta}_{UR} = \begin{pmatrix} a_0; & a_1; & a_2 \end{pmatrix}^T = \begin{pmatrix} 691,491; & -2,811; & 0,024 \end{pmatrix}^T$$

These estimates are biased and inconsistent due to the problem of endogeneity. We use the relationship between structural reduced coefficients (15) to calculate the correction factor (21) and write it in the matrix form [Babeshko 2018.]:

$$\begin{pmatrix} 1 & m_{31} & 0 \\ 0 & m_{32} & 1 \\ 0 & m_{33} & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} m_{11} \\ m_{12} \\ m_{13} \end{pmatrix}. \qquad (23)$$

To build the constraints matrix *H* and the vector of constants of constraints *r*, we use the OLS – estimates of the parameters of the reduced form (12):

[3]The number of restrictions in this model is equal to the number of columns of the matrix of reduced coefficients.

Table 3: Auxiliary Calculations

$X^T X$			$(X^T X)^{-1}$		
17	4267,02	5,018e+05	3,934e-01	-3,714e-03	2,025e-05
4267,02	1328280,22	1,607e+08	-3,714e-03	8,751e-05	-6,183e-07
5,018e+05	1,607e+08	1,973e+10	2,025e-05	-6,183e-07	4,572e-09
$(X^T X)^{-1} H^T$			$H(X^T X)^{-1} H^T$		
3,015e-01	6,315e-06	-1,883e-03	2,632e-01	-8,608e-07	-7,854e-04
-1,5495e-03	-2,9013e-07	4,438e-05	-8,608e-07	1,165e-09	-1,471e-04
4,949e-06	2,253e-09	-3,136e-07	-7,854e-04	-1,471e-04	2,250e-05
$V=(H(X^T X)^{-1} H^T)^{-1}$			$K=(X^T X)^{-1} H^T V$		
17	5,018e+05	3873,31	1	5,821e-11	-48,781
5,018e+05	1,973e+10	1,467e+08	-2,776e-17	-9,095e-13	1,972
3873,31	1,467e+08	1137312,55	0	1	-0,0074

$$H=\begin{pmatrix}1 & m_{31} & 0\\ 0 & m_{32} & 1\\ 0 & m_{33} & 0\end{pmatrix}=\begin{pmatrix}1 & 24,738 & 0\\ 0 & 0,004 & 1\\ 0 & 0,507 & 0\end{pmatrix},$$

$$r=\begin{pmatrix}\hat{m}_{11}\\ \hat{m}_{12}\\ \hat{m}_{13}\end{pmatrix}=\begin{pmatrix}666,389\\ 0,022\\ -2,698\end{pmatrix}.$$

Table **3** shows the values of the matrices used in formulas (15) - (18) in the order convenient for their calculation.

Auxiliary calculations and values of parameter estimates without restrictions (20) and with restrictions (19), are listed in Table **4**.

Comparison of the results given in the last column of Table **4** and the estimates of the parameters of models (10) and (16) shows that the estimates of the parameters of the structural equation of the SEM using the OLS with restrictions (5) on the structural parameters coincide with the ILS - estimates and 2SLS estimates, that solve the endogeneity problem.

Let us derive the formula for the autocovariance matrix of structural coefficients (19)

$$C_{RR}=Cov\left\{\hat{\beta}_{UR}+b,\hat{\beta}_{UR}+b\right\}=Cov\left\{\hat{\beta}_{UR},\hat{\beta}_{UR}\right\}+ 2Cov\left\{b,\hat{\beta}_{UR}\right\}+Cov\left\{b,b\right\}, \quad (24)$$

where $Cov\left\{\hat{\beta}_{UR},\hat{\beta}_{UR}\right\}=C_{\hat{\beta}\hat{\beta}}$ — autocovariance matrix of the estimates of the parameters (20) without restrictions,

$$Cov\left\{b,\hat{\beta}_{UR}\right\}=Cov\left\{K(H\hat{\beta}_{UR}-r),\hat{\beta}_{UR}\right\}=KHC_{\hat{\beta}\hat{\beta}}-KC_{r\hat{\beta}} =KHC_{\hat{\beta}\hat{\beta}}-KHC_{\hat{\beta}\hat{\beta}}=0,$$

where $C_{r\hat{\beta}}=Cov\left\{r,\hat{\beta}_{UR}\right\}=Cov\left\{H\hat{\beta}_{UR},\hat{\beta}_{UR}\right\}=HC_{\hat{\beta}\hat{\beta}}$,

$$Cov\{b,b\}=Cov\left\{K\left(H\hat{\beta}_{UR}-r\right),K\left(H\hat{\beta}_{UR}-r\right)\right\}=$$

$$=KHC_{\hat{\beta}\hat{\beta}}H^TK^T-KHC_{\hat{\beta}r}K^T-KC_{r\hat{\beta}}H^TK^T+KC_{rr}K^T=$$

$$=KHC_{\hat{\beta}\hat{\beta}}H^TK^T-KHC_{\hat{\beta}\hat{\beta}}H^TK^T-KHC_{\hat{\beta}\hat{\beta}}H^TK^T+KC_{rr}K^T=$$

$$-KHC_{\hat{\beta}\hat{\beta}}H^TK^T+KC_{rr}K^T,$$

as $HK=H(X^TX)^{-1}H^TV=H(X^TX)^{-1}H^T(H(X^TX)^{-1}H^T)^{-1}$

$$=H(X^TX)^{-1}H^T\left(H^T\right)^{-1}X^TXH^{-1}=I$$

Table 4: Calculation of Parameter Estimates with Constraints

$\hat{\beta}_{UR}$	$H\beta_{UR}$	$r-H\hat{\beta}_{UR}$	b	$\hat{\beta}_R$
1	2	3	4	5
691,491	621,958	44,431	106,530	798,021
-2,811	0,014	0,008	-2,510	-5,321
0,024	-1,425	-1,273	0,018	0,042

in the case of square nonsingular constraints matrix *H* (the number of equations in (15) is equal to the number of structural parameters).

Therefore,

$$\begin{aligned} C_{RR} &= C_{\hat{\beta}\hat{\beta}} - KHC_{\hat{\beta}\hat{\beta}}H^T K^T + KC_{rr}K^T = KC_{rr}K^T \\ &= H^{-1}C_{rr}\left(H^{-1}\right)^T \end{aligned} . \quad (25)$$

As it follows from (23), the autocovariance matrix of the estimates of the constraint's constants C_{rr} is the autocovariance matrix of parameter estimates for the first equation of the reduced form (12):

$$C_{rr} = \begin{pmatrix} 2071{,}586 & 0{,}0704 & -16{,}578 \\ 0{,}0704 & 2.158\text{e-}05 & -0{,}003 \\ -16{,}578 & -0{,}003 & 0{,}475 \end{pmatrix} . \quad (26)$$

And thus, the result of evaluation of the autocovariance matrix of parameter estimates with restrictions on the formula (25), considering (26) and the results given in Table **3**, coincides with its 2SLS-estimation:

$$C_{RR} = \begin{pmatrix} 4819{,}186 & -78{,}378 & 0{,}516 \\ -78{,}378 & 1{,}847 & -0{,}013 \\ 0{,}516 & -0{,}013 & 9{,}350\text{e-}05 \end{pmatrix} .$$

CONCLUSION

The article shows the equivalence of parameter estimates and their autocovariance matrices for the main methods of estimating the system of simultaneous equations, which solves the problem of endogeneity, using an empirical example. The estimation of the autocovariance matrix of ILS-parameter estimates is obtained as a result of the formalization of the 2SLS algorithm in the form of OLS with restrictions on the parameters. The result is demonstrated on the example of the cheese market model in Russia. The system of interdependent equations includes two endogenous variables (the volume of transactions and the equilibrium price) and two predefined variables (the current income of consumers and the lag value of the price). Both equations are precisely identifiable, which made it possible to apply ILS and perform a comparative analysis of the results.

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