On Poverty Traps and Equilibria in Growth Models

Olivier de La Grandville*

Department of Management Science and Engineering, Stanford University, 475 Via Ortega, Stanford, CA 94305, USA

Abstract: We show that, contrary to a widely spread error, when the savings and the population growth rates are constant, an unstable equilibrium cannot exist in a neoclassical model, because it would imply an increasing average productivity of capital and therefore a negative marginal productivity of labor. As a consequence, a poverty trap, a dire reality, cannot be explained by such an unstable equilibrium, nor cannot it be eliminated by a capital "big-push". We finally give necessary conditions for an economy to escape a poverty trap.

Keywords: Unstable equilibria, growth models.

1. INTRODUCTION

We usually associate recessions to the state of advanced economies whose income per person recedes and who suffer from increased unemployment. We thus tend to forget that poor countries have repeatedly shared the same fate, with one dramatic difference: those recessions are hitting poor and sometimes extremely poor populations.

The object of this paper is threefold: first we point out that in the time span 1960-2010 many countries have never made any progress toward escaping their poverty traps, while suffering severe, repeated recessions. Second, we show that the traditional theoretical approach to the possible escape from a poverty trap is seriously flawed. While the poverty trap is well understood and described as a stable equilibrium point in the well-known phase diagram of the capital-labor ratio, the traditional approach assumes the existence, higher on the capital-labor axis, of an unstable equilibrium. Such a construct – if it corresponded to reality – would imply that if some massive capital influx could be injected into that economy, income per person could increase toward a new, stable equilibrium. We will demonstrate that such an unstable equilibrium cannot exist, because it would imply a negative marginal productivity of labor. Finally, we will indicate the necessary conditions that should be met by countries intending to escape poverty traps.

2. THE BARE FACTS

We describe, in panels A to D, the evolution over period 1960 – 2010 of real GDP per person in 11 countries, together totaling more than 250 million people . To do so we relied on the Pen World Table set by Alan Heston, Robert Summers and Bettina Aten (Penn World Table Version 7.1, Center for International Comparisons of Production, Income and Prices at the University of Pennsylvania, July 2012). The numbers are expressed in 2005 US $ at purshasing power parity (Laspeyres index).

These dramatic pictures reveal that none of these countries experienced improvement in their standard of living, and that all of them suffered from repeated recessions. We now have to turn toward what traditional theory has to say about those poverty traps.

3. CORRECTING A SERIOUS MISTAKE

We now want to correct a widely spread error in the literature on growth and development as well as in macroeconomics. In neoclassical models, technological poverty traps are associated with the existence of a threshold level for the capital-labour ratio corresponding to a supposed unstable equilibrium. Should a capital "big push " enable the economy to exceed that level of the capital-labor ratio, income per person could either increase forever, or converge toward a new, higher equilibrium point.

Poverty traps, a dire reality, can well be explained within the framework of the neoclassical model. However, their very existence does not require an unstable equilibrium in the phase diagram of the capital-labour ratio. Even more importantly, we will show that such an unstable equilibrium cannot exist.When the savings and population growth rates are constant, an unstable equilibrium must result from – and indeed is always associated with – the assumption of an increasing average productivity of capital. But the neo-classical model does not support

*Address correspondence to this author at the Department of Management Science and Engineering, Stanford University, 475 Via Ortega, Stanford, CA 94305, USA; Tel: 1 650 724 7000; Fax: 1 650 723 1614;
E-mail: ola@stanford.edu
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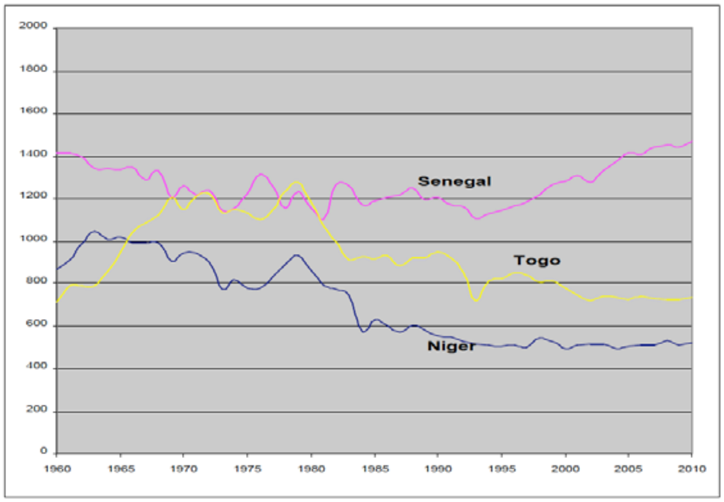
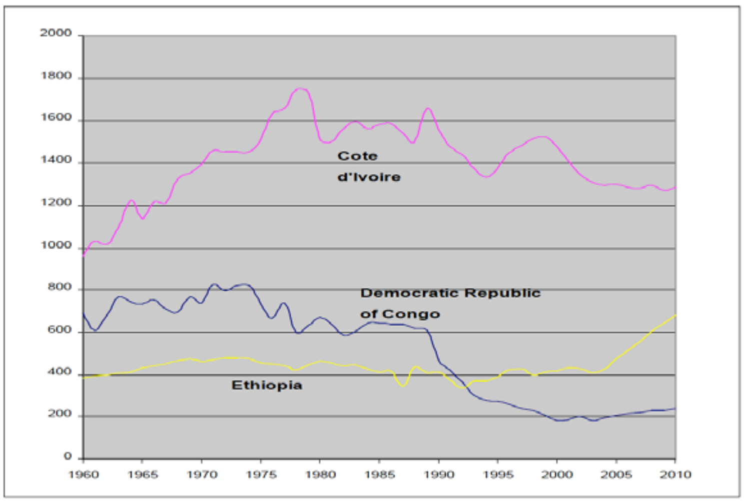
Panel A



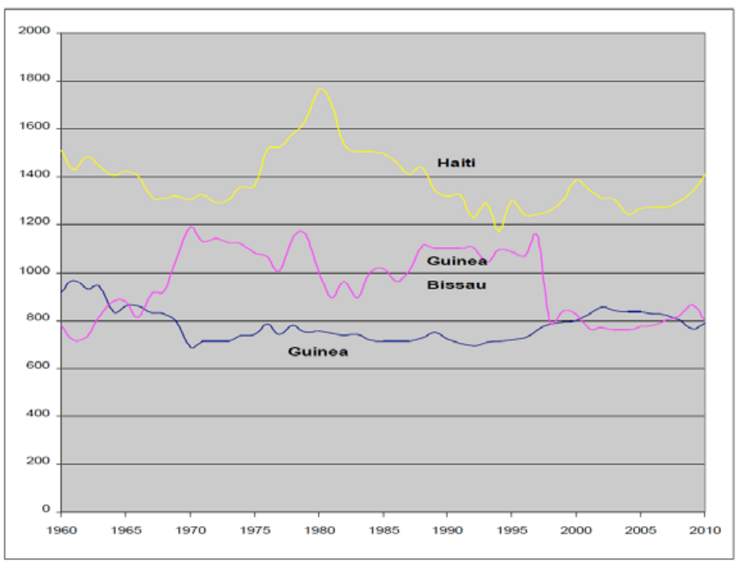
Real GDP per person; 2005 US $

Panel B



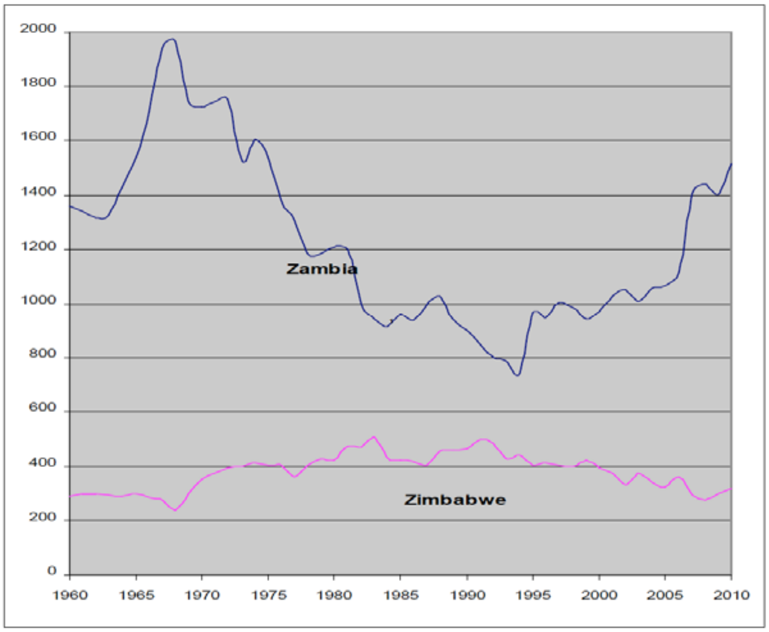
Real GDP per person; 2005 US $

Panel C



Real GDP per person; 2005 US $

Panel D



Real GDP per person; 2005 US $

that hypothesis because, as we will show, it would lead to a negative marginal productivity of labor.

The error we want to correct, consisting in assuming the existence of an unstable equilibrium due to a supposed increase of the average productivity of capital, has been made in innumerable papers and texts on growth and development; it has even spread into macroeconomics texts. The phase diagrams in Figures **1a** and **1b** below are so well-known that references are not necessary; such a correction thus seems long overdue.

3.1. Hypotheses

Production is assumed to be governed by a linearly homogeneous production function $Y = F(K,L)$. Y is national income, *net* of depreciation. The capital-labor ratio is denoted $K/L \equiv r$. Income per person, $Y/L \equiv y$, is then equal to $F(K/L,1) \equiv f(r)$. The marginal productivities, assumed to be positive, are then equal to equal to $\partial F/\partial K = f'(r)$ and $\partial F/\partial L = f(r) - rf'(r)$. We also assume the net savings-investment rate $s = \dot{K}/Y \equiv I/Y$ and the labour force growth rate $(\dot{L}/L = n)$ to be constant.

3.2. Stability Analysis

The equation of motion of the capital-labour ratio $r = K/L$ is usually expressed either in terms of its derivative $\dot{r}$, or in terms of its growth rate $\dot{r}/r$. So the basic differential equation reads either as

$$\dot{r} = sf(r) - nr, \quad (1)$$

or as

$$\frac{\dot{r}}{r} = s\frac{f(r)}{r} - n = sg(r) - n. \quad (2)$$

where $f(r)/r = Y/K = g(r)$ is the average productivity of capital; phase diagrams are drawn either in $(\dot{r}, r)$ space (Figure **1a**), or in $(\dot{r}/r, r)$ space (Figure **1b**). Countless expositions portray income per person and savings per person in such a way that over an interval of r the average productivity of capital $g(r)$ is increasing. In Figure **1a**, this is the open interval between points m and M of the curve $sf(r)$[1]; in Figure **1b**, it corresponds to the open interval between the local minimum m and the local maximum M of $sg(r)$. A so-called " unstable equilibrium" point r_2^* is thus generated, and considered as the threshold value of r that, if surpassed, would permit the economy to escape from the poverty trap r_1^*.

There are many ways to show that the average product of capital cannot be increasing because it

[1]See for instance Sachs, McArthur, Schmidt-Traub, Kruk, Bahadur, Faye and McCord (2004), in their otherwise excellent paper ìEnding Africaís Poverty Trap", Brookings Pa-pers on Economic Activity, 1, pp. 117-240. On page 125, in their Figure 2 entitled "Growth Model with Minimum Capital Stock Threshold", this interval exihibiting an increasing average productivity of capital stretches from the origin of the abscissa to a point midway between their threshold point k_T and the stable equilibrium value k_E.

would imply that the marginal product of labour is negative. Here are four of them.

(i) First, suppose that at some point r, $Y/K = g(r)$ is increasing with r; $g'(r) > 0$. By the definition of $g(r)$, Y is equal to

$$Y = Kg(r). \tag{3}$$

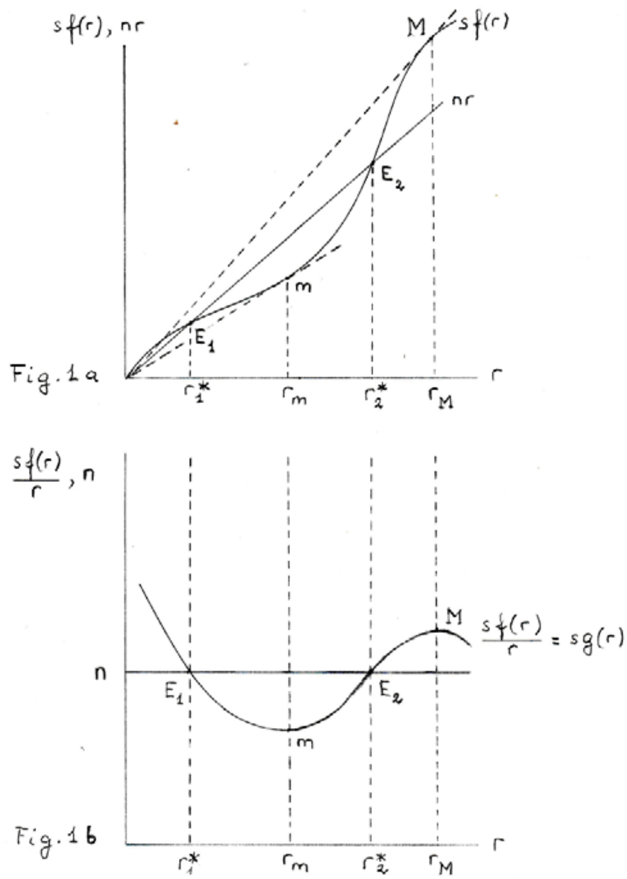


Figure 1: An unstable equilibrium E_2 would entail a negative marginal productivity of labor over the interval (m, M); it would be zero at points m and M. A poverty trap may exist at E_1, but it does not presuppose the existence of an unstable equilibrium point above it.

The marginal product of labor is then

$$\frac{\partial Y}{\partial L} = Kg'(r)\left(-\frac{K}{L^2}\right) = -r^2 g'(r), \tag{4}$$

which entails a *negative* marginal productivity of labour since $g'(r) > 0$.

(ii) From (3) the marginal product of capital is

$$\frac{\partial Y}{\partial K} = g(r) + rg'(r). \tag{5}$$

Since $g'(r) > 0$, $\partial Y/\partial K > g(r) = Y/K$; therefore $(\partial Y/\partial K)K/Y > 1$. From Euler's identity, $(\partial Y/\partial L)L/Y$ must then be negative, which implies $\partial Y/\partial L < 0$.

(iii) $g(r)$ is increasing, and $g'(r)$ is positive if and only if the marginal productivity of labour is negative: with $g(r) = f(r)/r$;

$$g'(r) = \frac{1}{r^2}[rf'(r) - f(r)] = -\frac{1}{r^2}\frac{\partial F}{\partial L} > 0 \Leftrightarrow \frac{\partial F}{\partial L} < 0. \tag{6}$$

(iv) Any per unit function such as Y/K is increasing if and only if the derivative $\partial Y/\partial K$ is larger than Y/K, i.e. if and only if the elasticity $(K/Y)\partial Y/\partial K$ is above one. By Euler's identity this is the case if and only if $\partial Y/\partial L < 0$.

It is possible to construct geometrically the real wage rate (assumed to be equal to $\partial F/\partial L$) at any point of the interval (m, M) and therefore to evaluate the order of magnitude of the error made. Consider the equilibrium point $E_2(r_2^*, sf(r_2^*))$ in Figure **1a** for instance. At that point draw the tangent to the curve $sf(r)$ (see Figure **2**).

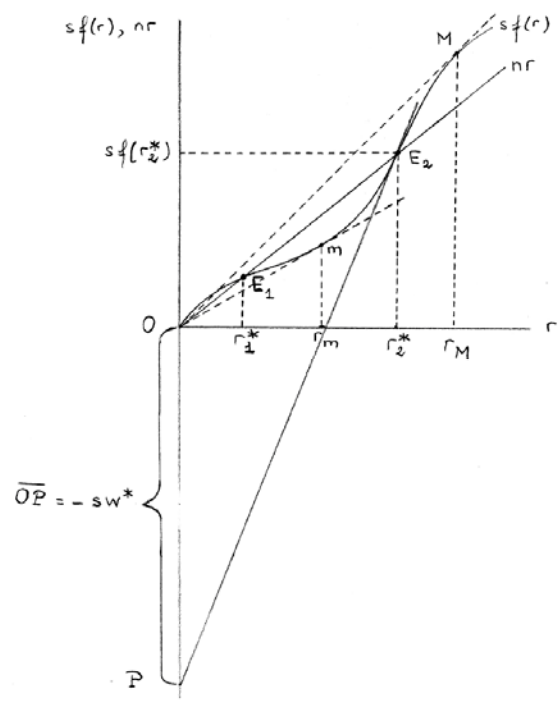


Figure 2: At any point of the interval (m, M) on the curve $sf(r)$ the marginal productivity of labor, equal to the real wage rate, would be negative by an amount easily determined. For instance, at equilibrium point E_2, it would be minus $1/s$ times the length $\overline{OP}$; the labor share would be $-\overline{OP}\ / sf(r_2)$ (about –2, or minus 200%) and the capital share $[sf(r_2^*) + \overline{OP}] / sf(r_2^*)$ (about +3).

This tangent cuts the ordinate at point P. The (positive) distance between $sf(r_2^*)$ and P is equal to the differential $sf'(r_2^*)r_2^*$. Due to $g'(r) > 0$ and (6), $s[f(r_2^*) - r_2^* f'(r_2^*)] < 0$; this is equal to *minus* the distance between O and P, denoted $-\overline{OP}$. Hence the pseudo equilibrium wage rate at point E_2 is equal to $w^*(r_2^*) = f(r_2^*) - r_2^* f'(r_2^*) = (-1/s)\,\overline{OP}$, a *negative* amount. Another way to gauge the error made is to observe from Figure **2** that the ratio of the wage rate to income per person, the labor share $w^*(r_2^*)/f(r_2^*)$, is about *minus* 2 (i.e about *minus* 200%). It implies a capital share $f'(r_2^*)r_2^*/f(r_2^*) = sf'(r_2^*)r_2^*/sf(r_2^*) = [\overline{OP} + sf(r_2^*)]/sf(r_2^*)$ – also read off from the diagram – equal to about plus 3.

The real wage rate, the real interest rate, and the shares of labour and capital in total income can also easily be constructed geometrically from a normally decreasing curve $Y/K = g(r)$; see Figure **3**. At any point (r,g), draw the tangent to the curve $g(r)$. The real rate of interest i, equal to the marginal productivity of capital, is $\frac{\partial F}{\partial K} = \frac{\partial}{\partial K}[Kg(r)] = g(r) + rg'(r)$. Let β (> 0) designate the length of the subtangent on the ordinate; then $g'(r) = -\beta/r$. Hence $\partial F/\partial K = i$ is the length $g(r) - \beta$ on the ordinate. From (4) the wage rate is $w = \frac{\partial F}{\partial L} = -r^2 g'(r) = r^2\beta/r = \beta r$. It is the rectangle with area βr in Figure **3**[2]. The share of capital in total income is $iK/Y = i/g$; the share of labour is β/g.

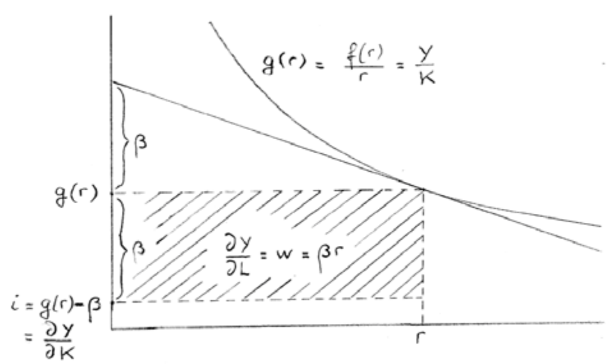


Figure 3: A normally decreasing average productivity of capital $Y/K = g(r)$ yields geometrically the interest rate i (length $g - \beta$), the real wage rate (area βr), the capital share (i/g) and the labour share (β/g).

[2]The fact that the rate of interest is measured by a vertical distance and the wage rate by an area in figures 1b and 3 should not be surprising: indeed, the measurement units on the ordinate are 1/time; those of the abscissa are \$/person; thus the units of an area are (\$/time)/person.

4. CONCLUSION: ESCAPING POVERTY TRAPS

Two conclusions need to be drawn. First, poverty traps can be explained in the neoclassical model of economic growth simply by the very existence of a stable equilibrium point $(r_1^*, sf(r_1^*))$ entailing a very low income per person. This can result from a low investment-savings rate s or from a low average productivity of labor $f(r)$ or, what is more likely to happen, from both causes simultaneously. But so-called technology poverty traps generated by an unstable equilibrium cannot exist, because the latter would imply a negative marginal productivity of labor.

Secondly, we can explain why, in the absence of a structural change in the production function itself, no capital increase in the form of a "big push", whatever its magnitude, can allow a poor country to escape a poverty trap. Needless to say, civilian peace and institutions, among which the separation of powers, are the backbone of any economic development. But from the sole technological point of view, with constant savings and population growth rates, only technical progress, eventually coupled with an increase in the elasticity of substitution[3], is capable of lifting an economy from such a trap.

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[3]In 1989, we made the conjecture that the remarkable growth observed in South Korea was due to an elasticity of substitution larger than in other countries (La Grandville, 1989). That conjecture then was successfully tested by Ky Hyang Yuhn (1991). For the role of the elasticity of substitution in enhancing growth, see also Klump and La Grandville (2000), La Grandville (2009) as well as La Grandville and Solow (2006), Thanh and Mach Nguyet (2008).

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