Study of the Tail Dependence Structure in Global Financial Markets Using Extreme Value Theory

Jian Wu[a,#], Zhengjun Zhang[b,*] and Yong Zhao[c,#]

[a]School of Economics and Management, Tsinghua University, Beijing 100084, China

[b]Department of Statistics, University of Wisconsin, Madison, WI 53706, USA

[c]Department of Finance, Investment and Banking, University of Wisconsin, Madison, WI 53706, USA

Abstract: The presence of tail dependencies invalidates the multivariate normality assumptions in portfolio risk management. The identification of tail (in)dependencies has drawn major attention in empirical financial studies. Yet it is still a challenging issue both theoretically and practically. Previous studies based on either a restrictive model or the null hypothesis of tail (perfect) dependence does not well describe or interpret extreme co-movements in financial markets. This paper examines tail dependence structures underlying a broad range of financial asset classes employing the newly developed tail quotient correlation coefficients. In theory, the original tail quotient correlation coefficient proposed in (Zhang 2008) is adapted to incorporate cases with varying data driven random thresholds. Our empirical results demonstrate different tail dependence structures underlying various global financial markets. Either omission or unanimous treatment of the tail dependence structures for different financial markets will lead to erroneous conclusions or suboptimal investment choices. The multivariate extreme value theory framework in this study has the potential to serve as an useful tool in exploiting arbitrage opportunities, optimizing asset allocations, and building robust risk management strategies.

Keywords: Tail dependence, Testing (tail) independence, Extreme value theory, Varying threshold values, Risk analysis.

1. INTRODUCTION

During the past years, the world economy has experienced unprecedented difficulties. The discrepancy between normality and reality has sparked a lot of awareness. Moreover, interests in extreme events in our society, reflected partially by flashy words such as six-sigma-events and black swan theory (Taleb 2007), have been popping up frequently in the Wall Street Journal and on CNBC. In the academic world, the study of Extreme Value Theory (EVT), dedicated to dealing with these “highly improbable” events, can be dated decades back. There are voluminous academic works on this topic, such as (Chamu 2005; de Haan and Resnick 1977; Embrechts, Kluppelberg and Mikosch 1999; Heffernan and Tawn 2004; Longin and Solnik 2001; Poon, Rockinger and Tawn 2004; Zhang 2008), to name a few, which deserve more attention given what has happened recently. Among topics on EVT, the study of tail dependence structures enjoys particular research interest due to its potential in solving practical financial problems, such as portfolio choices, hedging strategy enhancement, and credit risk analysis, among others. For instance, (Longin and Solnik 2001) used a logistic model (also called Gumbel copula) to test “extreme correlation” which is tail independence underlying the international equity markets. During the time the paper was written, Gumbel copula was a popular method and widely used in various applications. However, as pointed out in (Poon, Rockinger and Tawn 2004), the Gumbel copula was restrictive in the sense that it failed to reflect the wide variety of tail dependencies in reality and many other models. (Poon, Rockinger and Tawn 2004) used sub-models to measure tail dependencies and proposed several different tail independence structures to deal with the tail dependence underlying equity markets. They believe the international stock market returns tend to be asymptotically independent which, to us, is counterintuitive, thus puzzling, given the increasingly interrelated global financial markets. In this paper, address the limitations and puzzles among the aforementioned academic EVT work dealing with financial markets. First, we propose a better alternative test statistics, adapted from the tail quotient correlation coefficient proposed in the up-to-date work (Zhang 2008). Second, we apply the test statistics and simulation framework to beyond equity markets and into a broad range of financial markets. We start our discussion with a brief introduction of the fundamentals of EVT.

Extreme events, by definition, are events that rarely happen. Statistically speaking, they only appear in the tails of probability distributions. Extremal events in the

*Address corresponding to this author at the Department of Statistics, University of Wisconsin, Madison, WI 53706, USA; Tel: 1-608-262-2614; Fax: 1-608-262-0032; E-mail: zjz@stat.wisc.edu

[#]Co-Authors E-mails: wu.jason.j@gmail.com, yongzhaomd@gmail.com

JEL Classification: C12; C13; C4; C32; C53; G10.

financial world can take a variety of forms, such as the stock market crash as we experienced, major defaults in credit risk analysis, or the collapse of risky asset prices. In conventional statistical analysis, we treat these extremal events as outliers, and leave them out of our view most of the time. However, what happened recently in the financial market highlighted the importance of extremal events in asset pricing, portfolio choice, and risk management. The Pearson correlation, conventionally used as a dependence measure, neither differentiates between large and small observations, nor distinguishes between positive and negative observations. The failure of incorporating tail dependence underlying the financial markets that are actually asymptotically dependent will probably lead to erroneous conclusions and suboptimal investment choices.

Extreme Value Theory (EVT), in many cases, is the natural and most efficient way to study extremal events. EVT refers to a well established body of theories that are capable of predicting the occurrence of extremal events, outside the range of available data (Embrechts, Kluppelberg and Mikosch 1999). During the past decade, EVT has experienced fast development in the research areas of Actuarial Science, Risk Management, and Quality Engineering. As evidence, EVT is documented in a high volume of literature. For instance, (Diebold 1998; Neftci 2000; Zhang 2005) discuss EVT for the univariate cases, while (Bouye 2002; Buhl, Reich and Wegmann 2002; Heffernan and Tawn 2004; Mashal and Zeevi 2002; Smith 2003; Smith and Weissman 1996; Starica 1999; Starica 2000; Tawn 1990; Zhang and Smith 2004) extend the discussion of EVT to multivariate settings. (Longin and Solnik 2001) study the extreme correlation of the international equity market with the market directions employing EVT. The encyclopedic (Embrechts, Kluppelberg and Mikosch 1999) offers a comprehensive review of both EVT theory and its applications with an emphasis on the financial markets.

Among these researches, the multivariate version is particularly useful in the context of financial studies, since almost all financial applications involve more than one component, which can only be addressed through a multivariate framework. When all the constituents of a financial application, for instance, different pricing factors for a portfolio, reach their extremal levels, studying their relationships, namely, the extremal, asymptotic or tail dependence structures (these three terms are often used interchangeably), provides valuable information about the tail behavior of these constituents. It is well known that simultaneously modeling extreme observations across sections and time is a difficult task. Recently, (Chamu 2005; Zhang 2005; Zhang 2008; Zhang and Smith 2010) developed a rigorous estimation theory for a class of max-stable processes, Multivariate Maxima of Moving Maxima (M4) model proposed by (Smith and Weissman 1996), and applied this theory to model practical applications in the areas of Finance, Environmental Science and Telecommunications. More importantly, these articles lay out a coherent testing strategy, limiting distribution, and modeling framework.

More and more empirical evidences show that financial asset returns appear to be tail dependent. As a result, models lacking tail dependent specifications may not be efficient in risk management and portfolio choices, especially during the market recession or expansion time. Studying tail dependencies among financial risk variables has drawn a lot of attention in finance literature. However, testing tail independence has been considered as a difficult and interesting open problem. The present analysis is based on recent developments, with its focus on the tail independence testing and computation of a tail dependence measure. More specifically, the multivariate extreme value framework proposed by (Zhang 2005) is implemented to study the tail dependence structures among multiple risky financial asset classes through the proxy of the time series of several financial security indices. Besides empirical evidences of tail dependence, asymmetry is another research phenomenon in the finance literature. However, study of asymmetry is beyond the scope of the paper. Interested readers are referred to (Hong, Tu, and Zhou, 2007, Zheng, Shi, and Zhang, 2012).

The rest of the paper is organized as follows: In Section 2, we present the definition of the tail dependence measure, a brief introduction of the Generalized Extreme Value (GEV) distribution, the definition of the quotient correlation coefficient, and the design of the hypothesis test. As part of the theoretical contribution of this paper, a lemma and a theorem are documented in Section 2 as well, with the aim of extending the capability of the tail quotient correlation coefficient as proposed in (Zhang 2008) to addressing applications with varying (random) data driven thresholds. This section also contains an explanation of the data employed in this study. Section 3 characterizes the tail dependence structures underlying three financial markets, and the temporal evolution of a tail dependence measure. The conclusion remarks appear in Section 4. The Appendix offers the proof of the new theorem discussed in Section 2.

2. PRELIMINARIES

2.1. Methodology

In the literature, the univariate extreme value theory is by now well established as a statistical technique in data analysis and inference for extreme observations. There is also substantial writing on the theoretical background and statistical properties of the multivariate EVT framework. However, the multivariate EVT is less widely used in practice, especially in a higher (greater than 2) dimensional space, due to lack of efficient statistical models and estimation methods. A combination of max-stable processes and GARCH processes results in a new family of parametric model, capable of modeling cross-sectional tail dependence and tail dependence across time. Results from such models are well documented in (Smith 2003; Smith and Weissman 1996; Zhang and Smith 2004). For many applications in this literature, tail dependencies, and extremal co-movements between random variables are characterized by the tail dependence index as defined below.

Definition 2.1: Two identically distributed random variable pair (*X, Y*) are called asymptotically independent if

$$\lambda = \lim_{u \to x_F} P\left(Y > u \middle| X > u\right) = 0 \tag{1}$$

where $x_F = \sup\left\{x \in R : P\left(X \le x\right) < 1\right\}$ as studied in (Ledford and Tawn 2003; Schalter and Tawn 2003; Zhang and Smith 2004). λ is also called the bivariate upper tail dependence index which quantifies the degree of dependence of the bivariate tails. If $\lambda > 0$, the (*X, Y*) pair is deemed as asymptotically dependent.

Given this definition of tail dependence/ independence, the next question is how to estimate it, especially when the joint distribution of the random variables is unknown. (Zhang 2008) develops a new test statistic, namely, the gamma test statistic for the hypothesis test of tail dependence/independence. Employing the gamma test statistic, we test the null hypothesis $\lambda = 0$ against the alternative hypothesis $\lambda > 0$ without estimating λ directly. The simulation results in (Zhang 2008) also demonstrate that when the null hypothesis is tail independent (i.e. $\lambda = 0$), the gamma test is highly efficient. These new methodologies lay the theoretical foundation for this study. The main objective of this work is to apply these new theories to characterize the tail dependence structures underlying a broader range of financial asset classes.

Specifically, the modeling steps employed in this study are outlined as follows:

In the first step, the data set, which is chosen to be certain financial index time series, is fed into a Generalized Autoregressive Conditional Heteroscedasticity, or GARCH (Bollerslev 1986) model to filter out the volatilities embedded in these time series. After this GARCH filtering step, the resultant financial time series becomes approximately stationary, thereby far more amenable to the subsequent EVT estimation (Dias and Embrechts 2003; Engle 2002; McNeil and Frey 2000). The standardized time series are also called pseudo-observations. According to (Poon, Rockinger and Tawn 2004), the EVT analysis results are not sensitive to the parameter choice of the GARCH model.

The exceedances within the standardized time series that are defined as observations over a specific threshold are then fit to an extreme value distribution, because modeling of the exceedances over high thresholds is the widely accepted approach in the applications of extreme value theory. The only non-degenerate limit distribution of the exceedances over certain thresholds has been proven to be Generalized Pareto Distribution (GPD). The rigorous proof of this statement and the connection between GPD and Generalized Extreme Value distribution (GEV) can be found in (Pickands 1975). (Embrechts, Kluppelberg and Mikosch 1999) also offers detailed review and discussions on this issue. In this study, GEV is chosen as the extreme value distribution to fit the exceedances, since it is a more general approach. GEV is a convenient unifying representation of three types (Gumbel, Fréchet, and Weibull) of extreme value distributions, whose distribution functions take the following parametric form:

$$H(x) = \exp\left(-\left(1 + \xi \frac{x - \mu}{\psi}\right)_{+}^{-1/\xi}\right) \tag{2}$$

where μ is the location parameter, ψ >0 is the scale parameter, and ξ is the shape parameter. One point worth noting is that the GEV fitting process is applied to the positive and negative exceedance series, respectively. This approach enables the model to distinguish the tail dependences between the financial booming and distressing periods.

With the estimated parameters from the GEV fitting step, the transformation formula similar to those proposed in (Coles and Tawn 1994) are employed to convert the pseudo-observations to the unit Fréchet scale (μ =1, ψ =1, ξ =1), which is required by the subsequent hypothesis testing procedure. Values below the specific threshold are transformed based on their ranks.

After the data transformation step, we can employ the hypothesis test to explore whether the certain tail dependence structure is supported by the data. In this study, the tail quotient correlation coefficient and the corresponding gamma-test (or the chi-squared test in this paper) as presented in (Zhang 2008) are implemented to determine the tail dependence structures underlying the selected financial time series.

We need the following Lemma 2.2 and Theorem 2.3 from (Zhang 2008, Lemma 9.1 and Theorem 3.2) in deriving a new theorem, which allows varied threshold values.

Lemma 2.2: Suppose $X, X_1, X_2, \dots$, are positive random variables. Then $X_n \xrightarrow[P]{a.s.} X$ if and only if there are two sequences of random variables $\xi_1(n),\ \xi_2(n)$ such that $\xi_1(n) < X_n / X < \xi_2(n),\ n = 1,\ 2,\ \dots$, and $\xi_1(n) \xrightarrow[P]{a.s.} 1$, $\xi_2(n) \xrightarrow[P]{a.s.} 1$, as $n \to \infty$.

In this lemma, the notation *a.s*. stands for convergence with probability 1, and *P* stands for convergence in probability. We now illustrate how to identify the tail dependencies based on varying threshold values.

Let

$$\begin{pmatrix} X_1, & X_2, & \dots, & X_n \\ Y_1, & Y_2, & \dots, & Y_n \end{pmatrix} \tag{3}$$

be an independent array of unit Fréchet random variables. Now let $(U_i,\ Q_i),\ i = 1,\ \dots, n$ be a bivariate random sequence, where both U_i and Q_i are correlated and have support over $(0,\ u]$ for a high threshold value *u*. Let $X_{ui} = X_i I_{\{X_i > u\}} + U_i I_{\{X_i \le u\}}$, $Y_{ui} = Y_i I_{\{X_i > u\}} + Q_i I_{\{X_i \le u\}}$, $i = 1,\ \dots, n$. Then

$$\begin{pmatrix} X_{u1} \\ Y_{u1} \end{pmatrix}, \begin{pmatrix} X_{u2} \\ Y_{u2} \end{pmatrix}, \dots, \begin{pmatrix} X_{un} \\ Y_{un} \end{pmatrix} \tag{4}$$

is a bivariate random sequence drawn from two dependent random variable X_{ui} and Y_{ui}. Notice that and $Y_{ui} I_{\{Y_{ui} > u\}}$ $\left(= Y_i I_{\{Y_i > u\}}\right)$ are independent, but $X_{ui} I_{\{X_{ui} \le u\}}$ $\left(= U_i I_{\{X_i \le u\}}\right)$ and $Y_{ui} I_{\{Y_{ui} \le u\}}$ $\left(= Q_i I_{\{Y_i \le u\}}\right)$ are dependent. In fact, one can easily construct arbitrarily dependent structure for values below the threshold value of u .

Theorem 2.3: Suppose V_i and W_i, $i = 1, \dots, n$ are exceedance values (over the threshold u) in (4), i.e. $V_i \equiv \max(X_i - u, 0)$ and $W_i \equiv \max(Y_i - u, 0)$, and U_i and Q_i have the distribution $e^{1/u} e^{-1/x},\ 0 < x < u$. Define

$$q_{u,n} = \frac{\max_{i \le n} \left\{(u + W_i)/(u + V_i)\right\} + \max_{i \le n} \left\{(u + V_i)/(u + W_i)\right\} - 2}{\max_{i \le n} \left\{(u + W_i)/(u + V_i)\right\} \times \max_{i \le n} \left\{(u + V_i)/(u + W_i)\right\} - 1}, \tag{5}$$

where u is a positive threshold. Then the random variables $\max\limits_{i \le n}(u + W_i)/(u + V_i)$ and $\max\limits_{i \le n}(u + V_i)/(u + W_i)$ are tail independent, i.e.,

$$\lim_{n \to \infty} P\left(\frac{1}{n}\left[\max_{i \le n} \frac{u + W_i}{u + V_i} + 1\right] \le x, \frac{1}{n}\left[\max_{i \le n} \frac{u + V_i}{u + W_i} + 1\right] \le y\right) = e^{-(1 - e^{-1/u})/x - (1 - e^{-1/u})/y}, \tag{6}$$

Furthermore, as $n \to \infty$ the random variable $q_{u,n}$ is asymptotically chi-squared distributed, i.e.,

$$2n(1 - e^{-1/u}) q_{u,n} \xrightarrow{d} \chi_4^2, \tag{7}$$

where χ_4^2 is chi-squared distributed random variable with 4 degrees of freedom.

The limits of $q_{u,n}$ from a broad class of bivariate distributions are shown to be directly related to the tail dependence index. The value of $q_{u,n}$ is interpreted as the probability that one stock at least has a big price drop (negative return over u) given another stock already has a big price drop (negative return over u). In other words, this tail dependence quantity straightforwardly informs regulators, risk managers, and practitioners of the existence of extreme co-movements in the market. This dependence measure is both intuitively and practically appealing. The following example shows $q_{u,n}$ converges to a positive value with probability one.

Example 2.4: Suppose $Z_l,\ l=1,\dots L,$ are independent unit Frechét random variables with distribution function $F(x)=\exp(-1/x), x>0.$ Let $X=\max_{1\le l\le L}\alpha_{l1}Z_l$, $Y=\max_{1\le l\le L}\alpha_{l2}Z_l$, where $\alpha_{l1}>0$, $\sum_l \alpha_{li}=1, i=1,2.$ Let $c_1=\max_{l\le L}(\alpha_{l1}/\alpha_{l2})$ and $c_2=\max_{l\le L}(\alpha_{l2}/\alpha_{l1})$, then $P(X/Y\le c_1)=P(\max_l \alpha_{l1}Z_l<\max_l c_1\alpha_{l2}Z_l)=1$, and $P(Y/X\le c_2)=P(\max_l \alpha_{l2}Z_l<\max_l c_2\alpha_{l1}Z_l)=1$. Therefore, $\max_{i\le n}\{Y_i/X_i\}\xrightarrow{a.s.}c_1$, $\max_{i\le n}\{X_i/Y_i\}\xrightarrow{a.s.}c_2$, and $q_n\xrightarrow{a.s.}\dfrac{c_1+c_2-2}{c_1\times c_2-1}$, as $n\to\infty$.

As an illustration of the capability of the construction in Example 2.4, we show in Figure **1** the simulation results of various bivariate dependence structures and their comparison with both X and Y under different L choices from Example 2.4. It becomes obvious from Figure **1** that the construction in Example 2.4 is capable of capturing a variety of dependence structures.

Certainly, there are alternative extreme dependence measures, such as coefficient of tail dependence (Ledford and Tawn 1996) derived from a sub-model, in the extreme value literature. In the study of the asymptotic dependency of equity and fixed income securities, (Hartmann, Straetmans, and de Vries 2004) use the asymptotic distribution of Ledford and Tawn's coefficient of tail dependence to test the null hypothesis

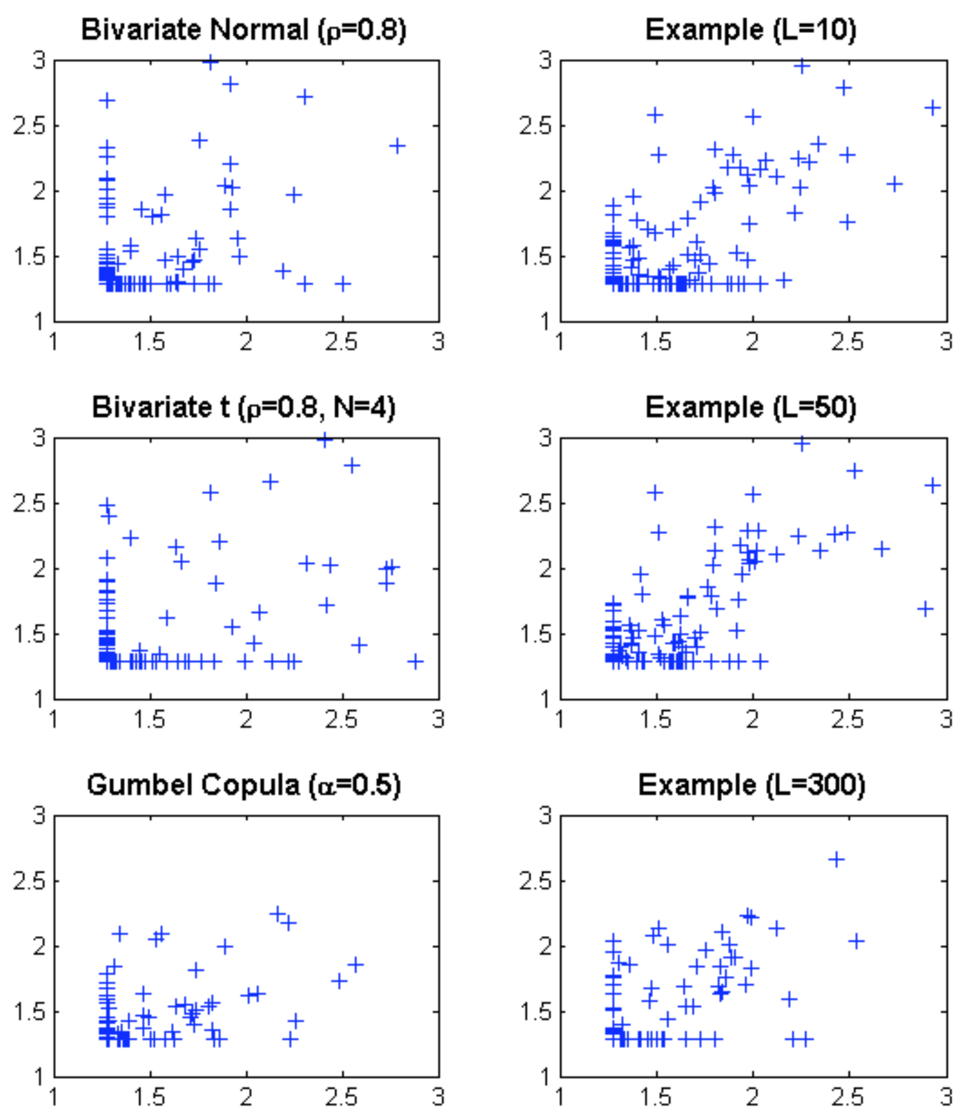

Figure 1: Comparison of different bivariate dependence structures with the construction in Example 2.4 under different choices of L. All random variables are in standard normal scales. In each subplot, we only plot the values above threshold 1.28, i.e., above 90% quantile.

of asymptotic dependence between two random variables. However it is not our purpose to compare those methods here. Our main goals are to calculate tail quotient correlation coefficients and to explore market behavior.

Notice that in practice we may need to choose a threshold value based on the observed values or transformed values. V_i and W_i may also be approximated by $\tilde{V}_i$ and $\tilde{W}_i$, respectively, where $\tilde{V}_i$ and $\tilde{W}_i$ are marginally transformed values based on a fitted parametric distribution or a non-parametric transformation method. We now present a new theorem, closely related to Theorem 2.3.

Theorem 2.5: With the established notations in Theorem 2.3, suppose $\tilde{V}_{i,n}$ and $\tilde{W}_{i,n}$ are exceedance values (over the threshold $u_n = u_n^* a_n$), where $\tilde{V}_{i,n} \xrightarrow[P]{a.s.} V_i$, $\tilde{W}_{i,n} \xrightarrow[P]{a.s.} W_i$, $u_n^* \xrightarrow[P]{a.s.} u, a_n \to \infty, a_n / n \to 0$ as $n \to \infty$. Define

$$\tilde{q}_n = \frac{\max_{i \le n}\left\{\left(u_n + \tilde{W}_{i,n}\right)/\left(u_n + \tilde{V}_{i,n}\right)\right\} + \max_{i \le n}\left\{\left(u_n + \tilde{W}_{i,n}\right)/\left(u_n + \tilde{V}_{i,n}\right)\right\} - 2}{\max_{i \le n}\left\{\left(u_n + \tilde{W}_{i,n}\right)/\left(u_n + \tilde{V}_{i,n}\right)\right\} \times \max_{i \le n}\left\{\left(u_n + \tilde{W}_{i,n}\right)/\left(u_n + \tilde{V}_{i,n}\right)\right\} - 1}. \quad (8)$$

Then as $n \to \infty$, the random variable $\tilde{q}_n$ is asymptotically chi-squared distributed, i.e.,

$$2n(1 - e^{-1/u_n})\tilde{q}_n \xrightarrow{d} \chi_4^2, \quad (9)$$

A proof of Theorem 2.4 is presented in Appendix.

Corollary 2.6: With the established notations in Theorems 2.3 and 2.4, suppose u is the pth percentile of unit Fréchet distribution. Suppose $X_{n,p}$ is the pth sample percentile of X_i s, and $Y_{n,p}$ is the pth sample percentile of Y_i s. Define $u_n = \min(X_{n,p}, Y_{n,p})$, then Equation (9) holds.

Equations (8) and (9) and Corollary 2.5 together constitute a test statistic, namely, the chi-squared test, which can be used to determine whether there is tail dependence between two random variables. The corresponding hypothesis test is designed as:

H0: X and Y are tail independent versus *H1*: X and Y are tail dependent.

When $2(1 - e^{-1/u_n})\tilde{q}_n > \zeta_\alpha$, *H0* is rejected in favor of *H1*, where ζ_α is the upper α th percentile of the chi-squared distribution with 4 degrees of freedom, which is the asymptotic distribution of the quotient statistic as in Eq. (8). The efficiency of the chi-squared test based on Theorem 2.3 has been illustrated in (Zhang 2008). The empirical power has been shown to be about 88%. In this study, the chi-squared test based on Corollary 2.5 is applied to each local window, which comprises 500 consecutive observations, and the local window will slide from the beginning to end of each time series to expose the temporal variation of the tail dependence underlying different financial markets. In each window, u is set to be u_n.

We argue that changing a fixed threshold u to a randomly varying threshold u_n makes V_i and W_i being dependent, and $\tilde{V}_{i,n}$ and $\tilde{W}_{i,n}$ being dependent as well. As sample sizes tend to infinity, the varying threshold u_n tends to infinity. We really deal with tail values. Each of those pair tail values tends to be independent, i.e. asymptotically independent. In other words, Eqn. (9) does test the null hypothesis of tail independence. We have not seen any other statistics possess this property. Based on this, it may be reasonable and safe to say that our work is the first to conceptually solve a tail independence testing problem.

2.2. Data

As mentioned in the introduction, a significance of this study is the extension of the financial asset classes under consideration beyond the equity assets into other asset categories, such as fixed income assets and commodity assets. The time series of multiple financial security indices will be employed as proxies for the corresponding asset classes to explore the tail dependence structures embedded in them.

Specifically, for the equity market, we focus on the tail dependence structures underlying three major European indices, namely, Financial Times index (FTSE 100), Cotation Assistee en Continu (CAC 40), and Deutscher Aktien IndeX (DAX), rather than the three U.S. indices (i.e. DJIA, S&P 500, and NASDAQ), which have been well studied in a number of previous works (Zhang 2005). The time horizon chosen for these equity indices ranges from November 26, 1990 to October 20, 2008, which is the maximum common time horizon for these three indices. In an effort to test the tail dependence structure across different financial asset classes, an equity index and a fixed income index are paired. The S&P 500 index and the yield on

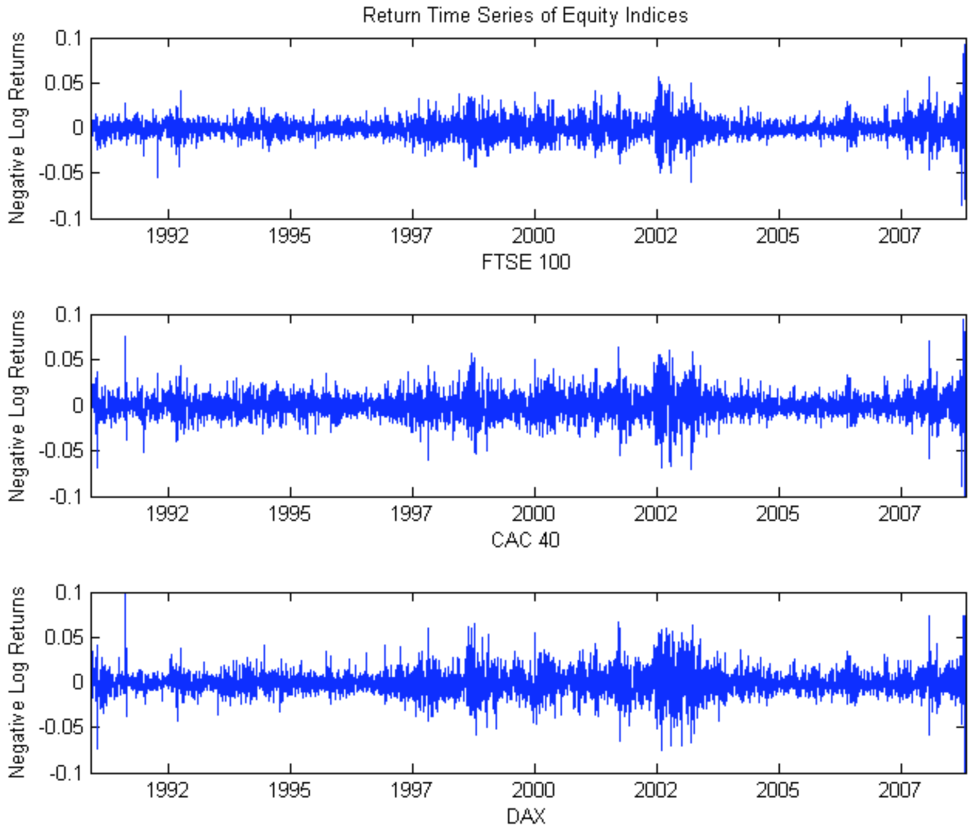

Figure 2: Negative logarithmic return time series of the three equity indices: FTSE 100, CAC 40, and DAX. Daily returns ranging from November 26, 1990 to October 20, 2008 are plotted.

the U.S. 10-year Treasury Note are used as the proxies for these two markets, respectively. Similarly, the maximum common time horizon for these two indices, which ranges from January 2, 1962 to October 20, 2008, is chosen for these two time series. As another test on the cross asset class tail dependence structure, the DJIA index (used to proxy the equity market) and the Dow Jones AIG Commodity index (used to proxy the commodity market) are paired for the tail dependence test. Based on the same rationale, January 3, 1991 to October 20, 2008 is determined to be the time horizon for these two time series.

For all of the security indices mentioned above, daily price data are retrieved from Yahoo! Finance for subsequent analysis in this study.

3. RESULTS

3.1. Data Processing and Transformation

Before the actual tail dependence modeling and testing step, the daily price time series are first converted to logarithmic return series, due to the statistically appealing properties associated with the return series (Cochrane 2005; Tsay 2005). As an illustration, the negative logarithmic returns of the three selected equity indices are plotted in Figure **2**. From this figure, we are able to identify extremal return observations, jumps in returns, and clustering in volatility from all three equity index return series.

As indicated in (Embrechts, Kluppelberg and Mikosch 1999), the EVT modeling of the tails of a distribution requires the observations to be stationary. It is well known that most financial return series exhibit certain degrees of autocorrelation, and more importantly, heteroscedasticity. In order to mitigate their effects on the subsequent EVT modeling, the raw return series are filtered by a GARCH(p,q) model. Here, we do not assume the financial time series under consideration are GARCH, rather we employ a GARCH process to model volatilities. From this GARCH filtering step, the conditional standard deviations embedded in these raw return time series are extracted. Then the original return time series are divided by estimated conditional standard deviations to obtain pseudo-observations. (Poon, Rockinger and Tawn 2004) points out that the tail dependence results are not sensitive to the choices of the volatility filters. As such, the results presented in this study are all based on a GARCH(1,1)

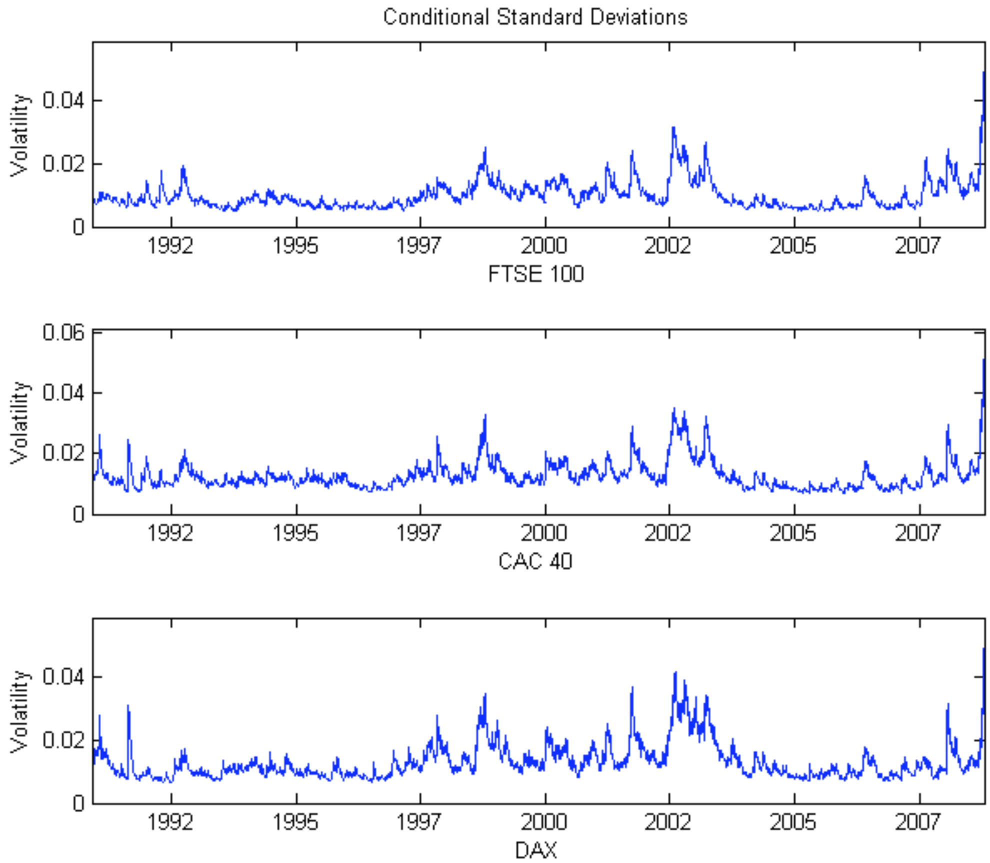

Figure 3: The conditional standard deviation time series corresponding to FTSE 100, CAC 40, and DAX equity indices from a GARCH(*1,1*) filtering process.

model fitting. Yet, more sophisticated GARCH models, such as heavy-tail GARCH models (Mikosch 2003; Nelson 1991), or GARCH models with weakly stationary residuals as studied in (Lee and Hansen 1994; Shinki and Zhang 2008), can be treated as useful extensions to this body of work. The GARCH(*1,1*) filtered conditional standard deviations present in all three equity index time series are shown in Figure **3**.

From Figure **3**, it becomes obvious that all return time series demonstrate variations in volatility to certain degree. The conditional standard deviation time series reminds us of several familiar instances when the financial markets experience excess volatilities, such as the 1997-1998 Russian credit crisis and Asian financial crisis, and the 2001-2002 tech-bubble. Additionally, all three time series in Figure **3** peak at the right end, which put the intensity of the financial turmoil that we are experiencing right now into historical perspective. The conditional standard deviation series are employed subsequently to standardize the raw return series. The devolatilized return series are also called pseudo-observations, and they have drawn attentions in a large number of financial data analysis, e.g. (Dias and Embrechts 2003; Engle 2002; McNeil and Frey 2000), to name a few. As will be shown shortly (Figures **5-7**) with the unit Fréchet transformation plots, one can immediately tell that the devolatilized time series are more stationary in comparison to their original counterparts, but the GARCH fitting process does not eliminate the persistency of the extremal jumps present in the return series. This result is consistent with that reported in (Poon, Rockinger and Tawn 2004), although heteroscedasticity is a major source of tail dependence, it cannot explain all the market co-movement on its own.

The next task is to fit the exceedance data within the pseudo-observation series over some high thresholds (u) to a generalized extreme value (GEV) distribution. In the empirical study, we choose the threshold u in a way to leave ten percent of the pseudo-observations above/below u, in order to carry out reasonable parameter estimation. The parametric formula of the GEV distribution as given in Eq. (2) is employed in a maximum likelihood estimation (MLE) to determine the three parameters, namely, the location parameter μ, the scale parameter ψ, and the shape parameter ξ, in the GEV distribution. As mentioned in the methodology section, the GEV fitting is applied to both the positive and negative pseudo-observation time series, separately. The estimated parameters from the GEV fitting are summarized in Tables **1** and **2** for the negative and positive exceedance series, respectively. From the last column of both Tables **1** and **2**, we notice that the shape parameter ξ is larger than zero. More importantly, the confidence intervals for all ξ estimations are positive, and do not include zero. This can be treated as strong evidence that the probability distributions of the underlying financial asset returns are heavy-tailed. Hence, the conventional thin-tailed distribution (e.g. normal distribution) based modeling approach turns out to be not appropriate.

In order to provide an intuitive assessment of the GEV fitting, the empirical Cumulative Distribution Function (CDF) of the positive tail exceedances within the pseudo-observation series of the CAC 40 equity index along with its CDF fitted from the GEV distribution are illustrated together in Fig Figure **4**. From this figure, we can tell that the fitted distribution closely follows the exceedance data.

In order to isolate dependence aspects from marginal distributional features, it is convenient to transform the pseudo-observation series to a standard marginal distribution. Theoretically, the pseudo-observation series can be transformed to any distribution, as long as the distribution function is continuous and strictly increasing. However, the hypothesis testing statistics and the corresponding asymptotic distribution as presented in Eqs. (7) and (8) require the data to be under the Fréchet scale. Hence, in this study, we transform the pseudo-observation series by the inverse of the unit Fréchet distribution function based on the three parameters estimated from the GEV fitting step. As mentioned before, the positive and negative pseudo-observation series are fitted into GEV distributions separately. Accordingly, they are transformed into the unit Fréchet distribution separately as well. The parameters used for this transformation step come from the different sets of parameter estimations reported in Tables **1** and **2**. With these parameters, the formula reported in (Coles and Tawn 1994) are employed for transformation. As an illustration of the transformation results, the standardized logarithmic return series, together with the transformed positive and negative exceedance series under the unit Fréchet scale, are presented in Figures **5-7** for the FTSE 100, CAC 40, and DAX equity indices, respectively.

Table 1: Estimations of Parameters from GEV Fitting of Standardized Negative Exceedance Series.

Indices	N_u	μ **(CI)**	ψ **(CI)**	ξ **(CI)**
FTSE 100	459	1.4853 [1.4525, 1.5180]	0.2850 [0.2572, 0.3159]	0.3979 [0.2757, 0.5201]
CAC 40	465	1.5122 [1.4813, 1.5431]	0.2836 [0.2581, 0.3117]	0.3385 [0.2369, 0.4402]
DAX	467	1.4638 [1.4350, 1.4926]	0.2584 [0.2323, 0.2873]	0.5064 [0.3887, 0.6241]

N_u is the number of exceedances used in the GEV fitting, and the threshold is chosen as $u=1.2$.

Table 2: Estimations of Parameters from GEV Fitting of Standardized Positive Exceedance Series

Indices	N_u	μ **(CI)**	ψ **(CI)**	ξ **(CI)**
FTSE 100	497	1.4360 [1.4128, 1.4591]	0.2190 [0.1995, 0.2404]	0.3695 [0.2686, 0.4705]
CAC 40	471	1.4367 [1.4118, 1.4616]	0.2271 [0.2061, 0.2501]	0.3712 [0.2638, 0.4785]
DAX	473	1.4245 [1.3995, 1.4496]	0.2222 [0.2011, 0.2455]	0.3759 [0.2582, 0.4936]

N_u is the number of exceedances used in the GEV fitting, and the threshold is chosen as $u=1.2$.

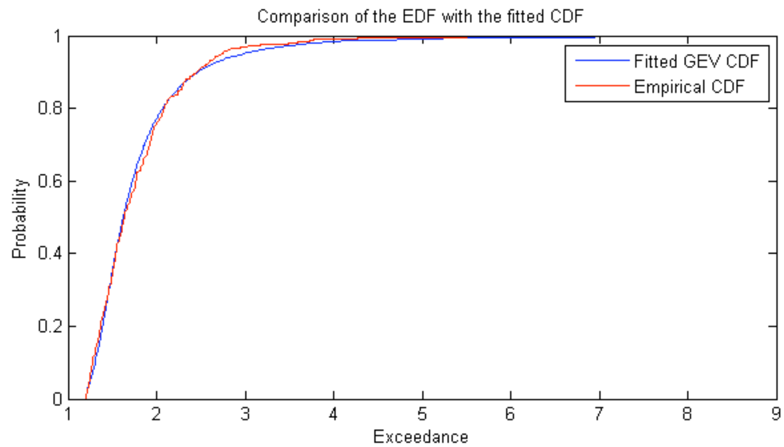

Figure 4: Comparison of the empirical CDF of the positive tail exceedances within the pseudo-observation series of the CAC 40 equity index with its CDF fitted from the GEV distribution.

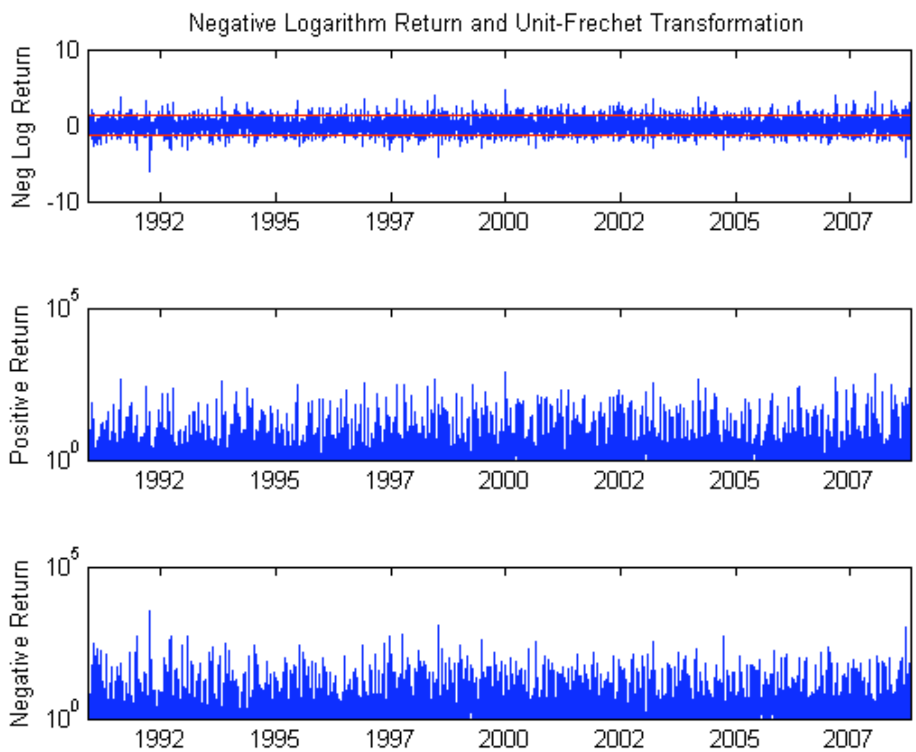

Figure 5: Standardized FTSE 100 negative logarithmic return series and the unit Fréchet transformation of both the positive and negative exceedance series.

Figures **5-7** reveal that the unit Fréchet transformation is a monotonic transformation of the exceedances, in the sense that the exceedances within all of the return time series retain their relative magnitudes before and after transformation. Direct observation of the series also makes it clear that among these three indices, the CAC 40 index looks more "similar" to the DAX index than to the FTSE 100 index. This is reflected through the fact that the exceedances with high magnitudes in both the positive and negative exceedances series appear at the same time on both the CAC 40 and DAX indices. Such a pattern is not apparent when we pair FTSE 100 with either CAC 40 or DAX indices. Hence, we have reason to conjecture that the CAC-DAX index pair will have higher tail dependence than both the FTSE-CAC and FTSE-DAX pairs. This intuition is further proven through the hypothesis test and the computation of the tail dependence index (λ) presented in the following sections.

The same data processing and transformation procedure performed above for the three equity indices are applied on all the time series employed in both the

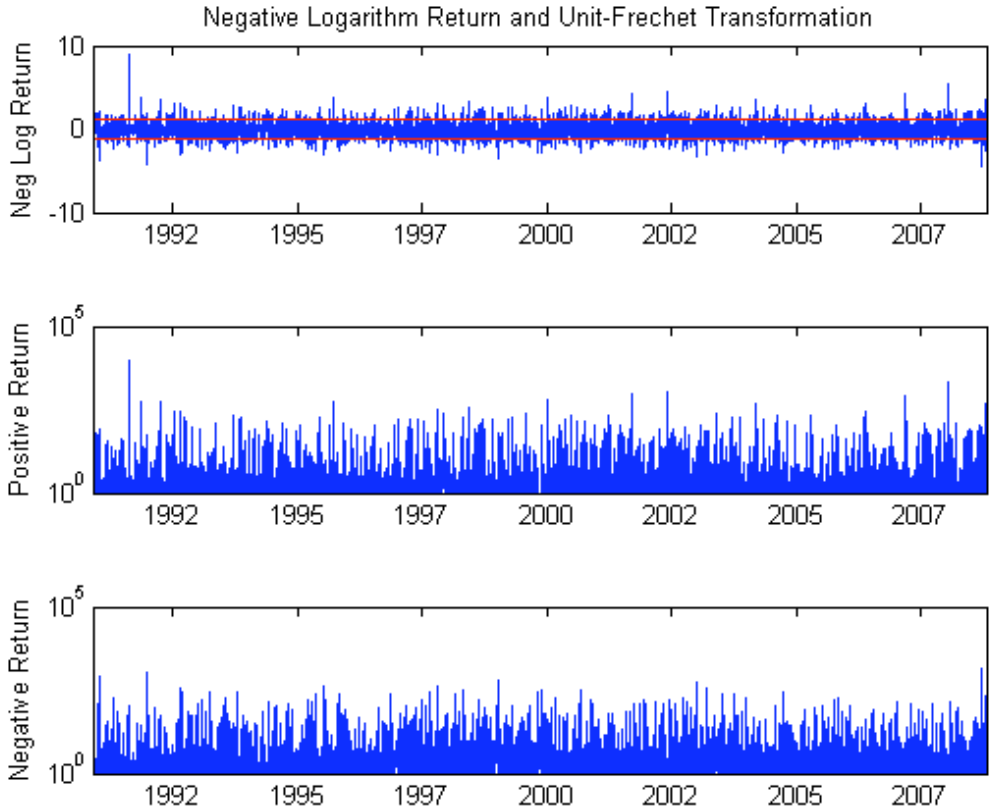

Figure 6: Standardized CAC 40 negative logarithmic return series and the unit Fréchet transformation of both its positive and negative exceedance series.

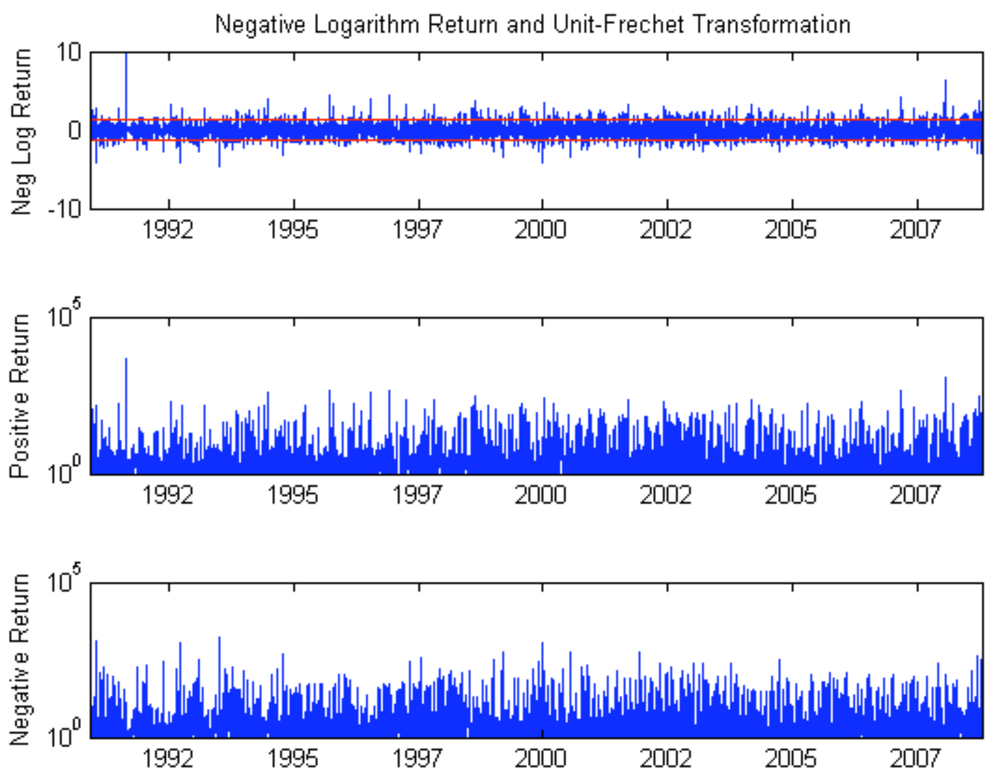

Figure 7: Standardized DAX negative logarithmic return series and the unit Fréchet transformation of both its positive and negative exceedance series.

equity-fixed income and the equity-commodity tail dependence tests. In Table **3** and **4**, the GEV fitting parameter estimations are reported for the negative and positive exceedance series, respectively. The plots for the pseudo-observation time series as well as their unit- Fréchet transformation plots are omitted because of their similarities to the results exhibited in Figures **5-7** for the equity indices. A noticeable result from both Tables **3** and **4** is that both the parameter estimation and the confidence intervals for the shape parameter ξ are away from zero, which is same as the heavy-tailed distribution we have observed on the equity markets. This testing result also implies the necessity of the EVT treatment of these financial markets.

3.2. Hypothesis Testing

After all pseudo-observation time series are transformed to the unit- Fréchet distribution, we can employ hypothesis test to determine whether certain tail-dependence structure is supported by the data. In this study, the tail quotient correlation coefficient and the corresponding chi-squared-test presented in our new Theorem 2.4 are applied for the hypothesis testing task. The subsets of the pseudo-observation data are used to control the Type I error. According to this scheme, 500 consecutive pseudo-observation points are sampled from all of the financial time series, and a full enumeration local window scheme is employed to study the temporal variation of the tail dependence structure as well. In each test, we use the 95th percentiles of the data as the threshold value. Thus, the test significance level is chosen as $\alpha = 0.05$. The test results are summarized in Table **5** for the three selected equity indices. In this table, Index Pairs column indicates the two pseudo-observation time series paired for the hypothesis test, which plays the role of the V_i and W_i series in Eq. (5) or the $\tilde{V}_i$ and $\tilde{W}_i$ series in Eq. (7). For instance, (FTSEn, CACp) means that the test results are reported by calculating the quotient correlation coefficient with its sample coming from the negative exceedance series of the FTSE 100 index and the positive exceedance series of the CAC 40 index. Other testing pairs are defined similarly. In

Table 3: Estimations of Parameters from GEV Fitting of Standardized Negative Exceedance Series

Indices	N_u	μ (CI)	ψ (CI)	ξ (CI)
S&P 500	1214	1.4495	0.2562	0.5296
		[1.4316, 1.4674]	[0.2395, 0.2740]	[0.4540, 0.6053]
10-Year Note	1073	1.4758	0.2874	0.5504
		[1.4543, 1.4973]	[0.2671, 0.3091]	[0.4672, 0.6335]
DJIA	453	1.4401	0.2619	0.6140
		[1.4098, 1.4704]	[0.2330, 0.2944]	[0.4816, 0.7464]
DJ-AIG	459	1.4767	0.2752	0.4546
		[1.4458, 1.5076]	[0.2480, 0.3054]	[0.3380, 0. 5712]

N_u is the number of exceedances used in the GEV fitting, and the threshold is chosen as $u = 1.2$.

Table 4: Estimations of Parameters from GEV Fitting of Standardized Positive Exceedance Series

Indices	N_u	μ (CI)	ψ (CI)	ξ (CI)
S&P 500	1228	1.4563	0.2478	0.4056
		[1.4392, 1.4733]	[0.2329, 0.2637]	[0.3347, 0.4766]
10-Year Note	1161	1.4840	0.2943	0.4997
		[1.4626, 1.5054]	[0.2747, 0.3153]	[0.4186, 0.5809]
DJIA	490	1.4497	0.2420	0.3679
		[1.4227, 1.4767]	[0.2193, 0.2670]	[0.2498, 0.4861]
DJ-AIG	476	1.4544	0.2442	0.4427
		[1.4276, 1.4812]	[0.2207, 0.2701]	[0.3306, 0.5548]

N_u is the number of exceedances used in the GEV fitting, and the threshold is chosen as $u = 1.2$.

Table **5**, the Rejection Rate column reports the percentage of rejecting H_0, when we slide the local window along the time series.

Table 5: The Hypothesis Test Results for the Tail Dependence Structure Underlying the Three Selected Equity Indices

Index Pairs	Rejection Rate
(FTSEp, CACp)	0.4811
(FTSEn, CACn)	0.3141
(FTSEn, CACp)	0
(FTSEp, CACn)	0
(FTSEp, DAXp)	0.5302
(FTSEn, DAXn)	0.3084
(FTSEn, DAXp)	0.011
(FTSEp, DAXn)	0.0588
(CACp, DAXp)	0.6153
(CACn, DAXn)	0.4015
(CACn, DAXp)	0
(CACp, DAXn)	0.0428

In the discussion that follows, we define the exceedance pair consisting of the constituent time series with the same sign (i.e. positive v.s. positive or negative v.s. negative) to be same-sign-pair and the pair including the time series with the opposite signs (i.e. positive v.s. negative or negative v.s. positive) to be opposite-sign-pair. The hypothesis test results summarized in Table **5** suggest that the degree of extremal co-movements in the same-sign-pairs with their constituent time series coming from different equity indices (for instance, FTSEp-CACp or CACn-DAXn), are high in general, as indicated by the high rejection rates for the corresponding pairs. A larger rejection rate means that the null hypothesis of the chi-squared test is rejected over a bigger portion of the time horizon. On the contrary, the opposite-sign-pairs of different equity indices tend to move independently in the tail parts no matter whether the two constituent exceedance time series come from the same index or from different indices. This conclusion can be drawn from the small rejection rates reported for these pairs in Table **5**. We can see that the rejection rates for all the opposite-sign-pairs are smaller in magnitudes when compared with the rejection rates for the same-sign-pairs. The test results for several opposite-sign-pairs are even zero, meaning nowhere on the time series exhibits tail dependence between the two underlying exceedance series. Detailed observations of the results in Table **5** also reveal that the degree of tail dependence between the CAC 40 and DAX indices is higher than that either between FTSE 100-CAC 40 pair or between FTSE 100-DAX pair, as indicated by the higher rejection rates for the same-sign-pairs consisting of the CAC 40 and DAX indices. This result is consistent with our conjecture by just looking at the pseudo-observation time series illustrated in Figures **5-7**.

Following the same hypothesis testing procedure, the tail dependence structures underlying the equity-fixed income market pair, as well as the equity-commodity market pair are explored. The hypothesis test results are summarized in Table **6** and **7**, respectively.

Table 6: The Hypothesis Test Results for the Tail Dependence Structure Underlying the Equity and Fixed Income Markets

Index Pairs	Rejection Rate
(SP500p, 10YEARp)	0.0686
(SP500n, 10YEARn)	0.0744
(SP500n, 10YEARp)	0.1841
(SP500p, 10YEARn)	0.0939

Table 7: The Hypothesis Test Results for the Tail Dependence Structure Underlying the Equity and Commodity Markets

Index Pairs	Rejection Rate
(DJIAp, DJAIGp)	0.0579
(DJIAn, DJAIGn)	0.0045
(DJIAn, DJAIGp)	0.0961
(DJIAp, DJAIGn)	0.0642

The hypothesis testing results presented in Tables **6-7** demonstrate the following tail dependence structures across different asset classes. The results in Table **6** expose the tail dependence structure between the S&P 500 index, which is used as a proxy for the equity market, and the 10-Year Treasury Note yield, which is used as a proxy for the fixed income market. In contrast with the tail dependence structure underlying the equity markets, which are reported in Table **5**, the tail dependence between the equity and the fixed income exceedance series is low in general, which is reflected through the low level of rejection rates. However, rejection rates for the opposite-sign-pairs are higher than those for the same-sign-pairs, which mean that the extremal returns of opposite signs have higher co-movement tendency. This different tail dependence structure can possibly be explained by the "fly to quality" effect that has long been observed between the equity and the fixed income markets. This conjecture can be further supported by the fact that the rejection rate for the (SP500n, 10YEARp) pair is the highest among the rejection rates for all pairs between these two markets. This means that when the equity market experiences extremal negative returns, the 10-Year Treasury Note market will experience extremal positive returns with relatively higher probabilities. The interaction between the recent historical losses on the U.S. equity markets and the unprecedented gains on the U.S. Treasury markets is powerful evidence of this tail dependence structure.

The results presented in Table **7** group the tail dependence structure between the equity and the commodity markets and underlying the equity and the fixed income markets into the same camp. The tail dependence between the equity market and the commodity market is low in general, although not as significant as the structure underlying the equity and the fixed-income markets. The opposite-sign-pairs from the equity and the commodity markets also demonstrate higher tail dependences than the same-sign-pairs do.

3.3. Tail Dependence Index

As a direct measure to quantify the tail dependence, the tail dependence index as studied in (Ledford and Tawn 2003; Schalter and Tawn 2003; Zhang and Smith 2004) is discovered in this section. The formula of the tail dependence index is given in Eq. (1), and the index computations are summarized in Table **8**.

From Table **8**, we can see that the empirical tail dependence index results are consistent with the hypothesis testing results reported in Tables **5-7**. The tail dependence index of the same-sign-pairs from the equity market is the highest among all exceedance pairs. The opposite-sign-pairs from the equity market exhibit the least level of tail dependence, as reflected by their close-to-zero M.E.D.I. The opposite-sign-pairs for the equity-fixed income markets and the equity-commodity markets exhibit higher M.E.D.I. than their

Table 8: The Tail Dependence Index for Different Financial Asset Classes

Index Pairs	M.E.D.I	Std. Dev.
(FTSEp, CACp)	0.6021	0.1137
(FTSEn, CACn)	0.5289	0.1390
(FTSEn, CACp)	0	0
(FTSEp, CACn)	0	0
(FTSEp, DAXp)	0.4880	0.1114
(FTSEn, DAXn)	0.4277	0.1092
(FTSEn, DAXp)	0.0099	0.0139
(FTSEp, DAXn)	0.0056	0.0097
(CACp, DAXp)	0.5645	0.1185
(CACn, DAXn)	0.5189	0.1400
(CACn, DAXp)	0.0042	0.0073
(CACp, DAXn)	0.0097	0.0131
(SP500p, 10YEARp)	0.0973	0.0955
(SP500n, 10YEARn)	0.0786	0.0953
(SP500n, 10YEARp)	0.2007	0.0907
(SP500p, 10YEARn)	0.2220	0.1115
(DJIAp, DJAIGp)	0.0996	0.0544
(DJIAn, DJAIGn)	0.1245	0.0424
(DJIAn, DJAIGp)	0.1146	0.0359
(DJIAp, DJAIGn)	0.1107	0.0431

In this table, M.E.D.I. stands for the average empirical estimation of the tail dependence index. Std. Dev. represents the sample standard deviation of the tail dependence index series.

same-sign-pair counterparts. This phenomenon is more apparent between the equity and the fixed-income markets. In this sense, the tail quotient correlation coefficient is indeed a tail dependence, i.e., a conditional probability at upper tails.

Besides the average empirical tail dependence index reported in Table **8**, the time series of the tail dependence index are also illustrated in Figures **8-10** employing the full local window enumeration scheme, in order to further expose the temporal evolution of the tail dependence structures underlying different financial markets.

From Figure **8**, we see that the tail dependence for the same-sign-pairs from the equity market is generally

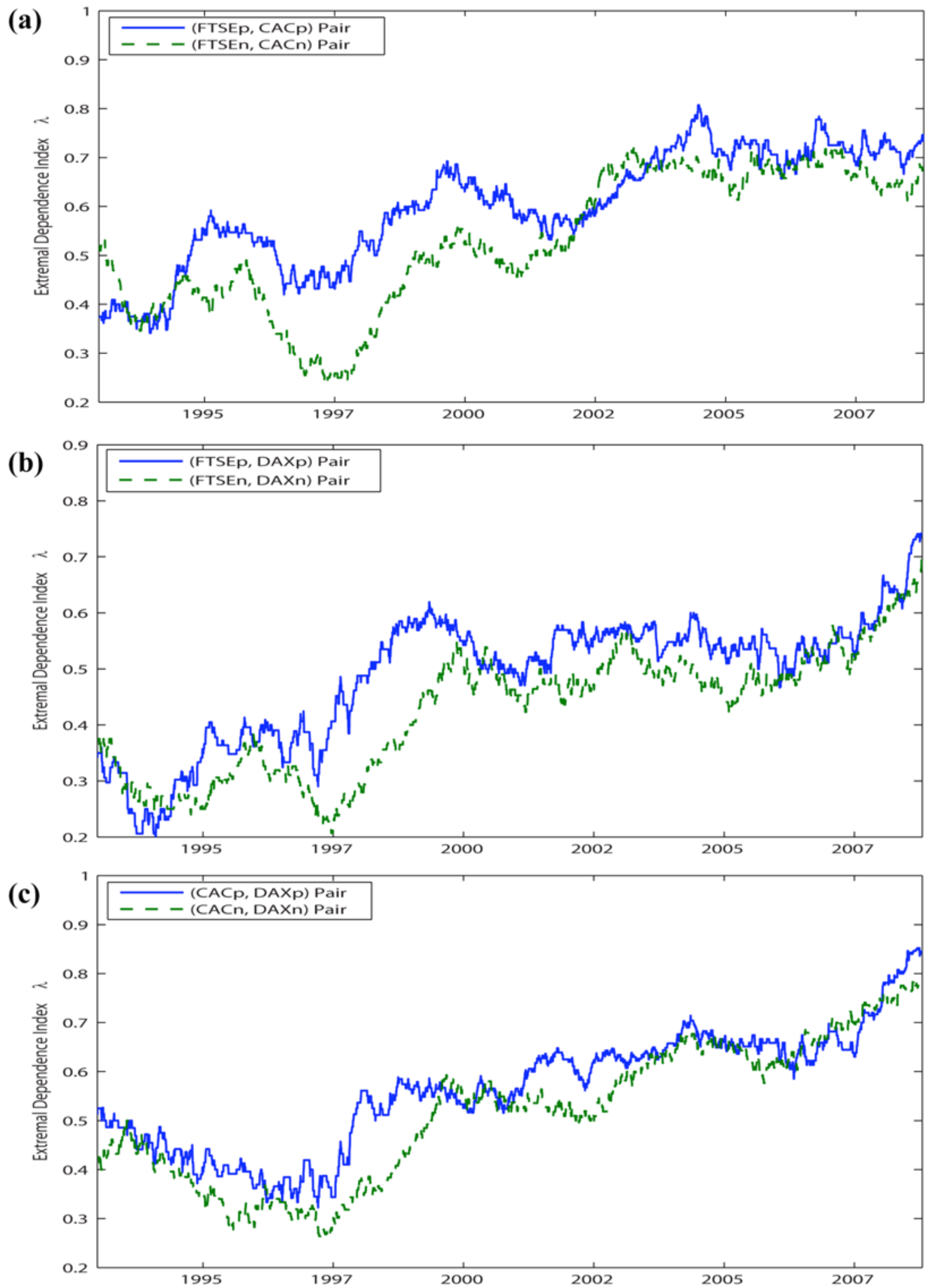

Figure 8: Time series of the empirical tail dependence index λ for the same-sign-pairs between (**a**) FTSE 100 and CAC 40 indices; (**b**) FTSE 100 and DAX indices; (**c**) CAC 40 and DAX indices.

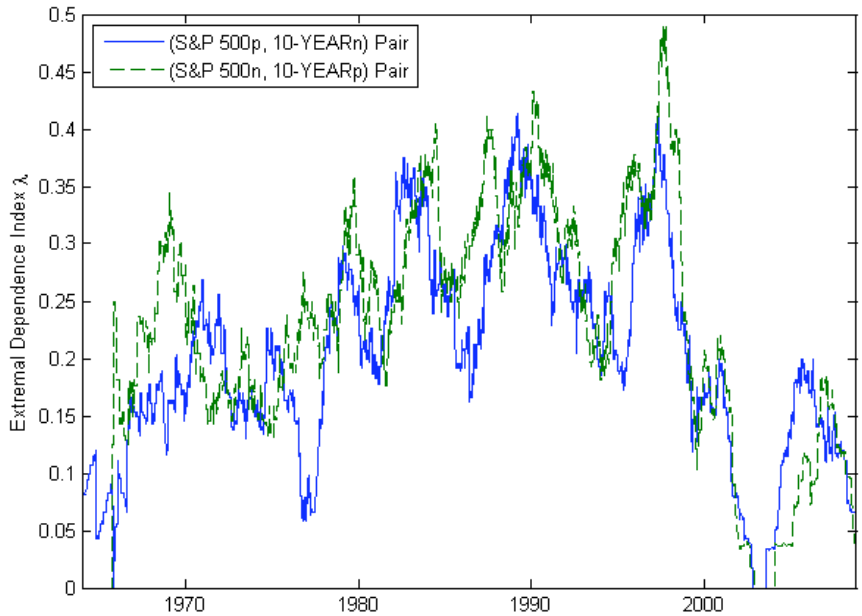

Figure 9: Time series of the empirical tail dependence index λ for the opposite-sign-pairs between the S&P 500 equity index and the 10-Year Treasury Note yield.

high, as reflected by the high magnitude of λ. This is consistent with the hypothesis testing results reported in the previous sections. Moreover, these time series tell us more about the temporal evolution of λ. As illustrated from the three subplots in Figure **8**, all λ series apparently exhibit upward trends. Since λ is a tail dependence measure, this upward trend means that the degrees of tail dependence underlying these equity markets increase over the testing period. The results plotted in Figure **8** are consistent with the tail dependence testing results for the European equity markets reported in (Poon, Rockinger and Tawn 2004), but in a more intuitive fashion. According to (Poon, Rockinger and Tawn 2004), the presence of tail dependence will render the traditional way of risk evaluation and management inefficient, if not invalid. If a model fails to incorporate the tail dependence underlying certain financial markets when they are actually tail dependent, it will probably overlook risk factors, hence underestimating the associated risks. When the degree of tail dependence increases, which is represented by the upward-trended λ series, the degree of erroneous risk evaluation and management also increases accordingly. As the mis-estimation and mismanagement accumulates, may it lead to a new round of turmoil on the financial markets?

As illustrated in Figure **9**, the time series of the tail dependence index for the opposite-sign-pair between the equity and the fixed-income markets exhibits a different pattern from the results illustrated in Figure **8**. Prior to 1997, the tail dependence index between the S&P 500 and the 10-year Treasury Note markets has a clear upward trend, meaning the degree of tail dependence between these two markets increases over that period of time. After 1998, the tail dependence index experiences a sharp drop down to zero, and then continues with an oscillatory increasing pattern during recent years. The tail dependence index for the opposite-sign-pair between the equity and the commodity markets, as plotted in Figure **10**, exhibits yet another different pattern. Neither series shows clear directional trend, rather demonstrating an oscillatory pattern. In addition, as opposed to the results shown in Figures **8** and **9**, where the two series in each figure show high correlations, the two series in Figure **10** present an obvious lag phenomenon. This pattern is especially prominent over the 1990 to 2007 period. During this period, it seems that the tail dependence index series for the (DJIAn, DJAIGp) pair lags four years behind that for the (DJIAp, DJAIGn) pair. This lag phenomenon may manifest the economic cyclic period underlying these two markets.

The study in this section extends the research in the similar studies, e.g. (Longin and Solnik 2001; Poon, Rockinger and Tawn 2004), which only place their focuses on the tail dependence structure underlying different equity markets. Both the hypothesis testing and the tail dependence index results support the conclusion that the tail dependence structures underlying different types of financial markets, e.g. equity v.s. fixed-income market or equity v.s. commodity market, are clearly different from the

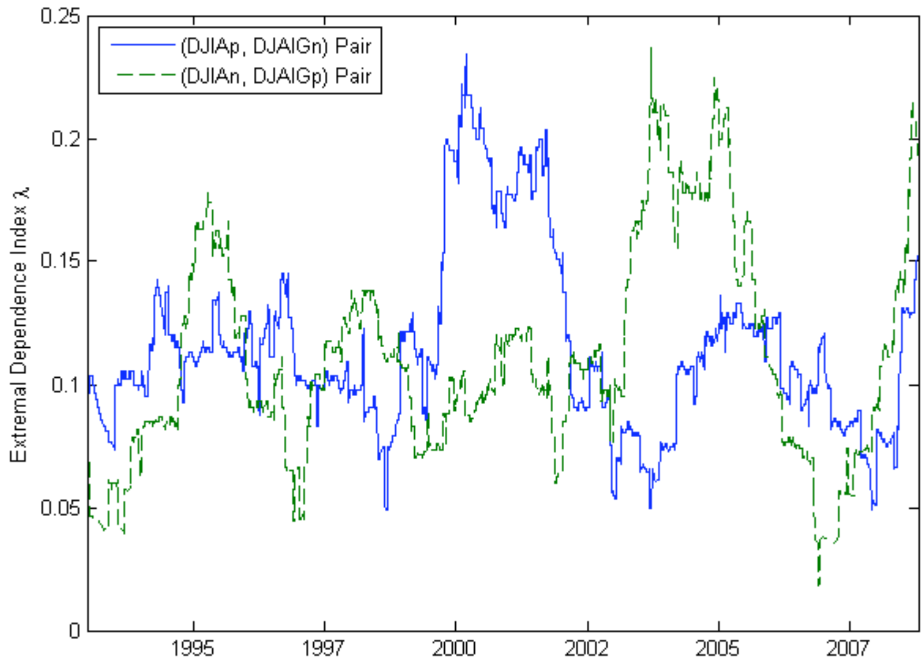

Figure 10: Time series of the empirical tail dependence index λ for the opposite-sign-pairs between the DJIA equity and the DJAIG commodity indices.

structure underlying the equity v.s. equity market. When we try to apply the tail dependence results to the practical scenarios as proposed in (Poon, Rockinger and Tawn 2004), namely, the portfolio choice, the Sharpe Ratio sharpening, the hedging strategy adjustment, the complex option valuation, and the credit risk analysis, it is important to be aware of the different tail dependence structures underlying these financial markets. Given an investment universe spanned by multiple financial asset classes, the recognition of the difference in tail dependence structures will lead to better informed and potentially optimal asset allocation and risk management decisions.

4. CONCLUSION

In this study, a multivariate extreme value framework is implemented, and the Extreme Value Theory (EVT) is employed to characterize the tail dependence structures for various financial asset classes. The EVT framework implemented in this study enables us to explore the widespread tail dependence phenomenon on the financial markets, which has been overlooked in the finance literature (Poon, Rockinger and Tawn 2004). Such omission may lead to erroneous estimation of market risks, or suboptimal asset allocations. As to the specific multivariate EVT model, a newly developed test-statistic, namely, the tail quotient correlation coefficient, as well as the associated chi-squared testing procedure are implemented to efficiently find the tail dependence structures underlying various financial markets and their temporal evolutions.

Another contribution of this study is that it overcomes the limitations of the previous similar researches, such as (Longin and Solnik 2001; Poon, Rockinger and Tawn 2004), which use sub-models in measuring tail dependencies and only deal with the tail dependence underlying the equity markets. Due to such limitations, the above research is silent about the tail dependence structures among other asset classes, hence is less helpful in addressing the needs of multi-strategy investment vehicles, whose investment universe easily spans beyond the equity world. The tail quotient correlation coefficients are defined for only tail values and have direct and intuitive tail probability interpretations. Besides the equity market, this study expands its scope of researching the tail dependence structure into non-equity financial markets, such as the fixed-income security market as well as the commodity market. From both the hypothesis testing and the tail dependence index computation results presented in Section 3, we can see clearly that the tail dependence structures underlying different financial markets are not unanimous. Treating the tail dependences for various financial markets equally will probably lead to erroneous conclusions and suboptimal investment choices.

The multivariate EVT framework, the statistical testing method, as well as the tail dependence measure implemented in this work can serve as a useful tool in exploiting the innovative EVT based arbitrage opportunities and building robust risk management strategies within a certain asset class and across different asset classes.

APPENDIX

A proof of Theorem 2.4 is provided as follows. By Lemma 2.2, there exist $\xi_j(n)>0,\ \ \xi_j(n)\xrightarrow{a.s.}1,\ j=1,2,\ldots,6,$ as $n\to\infty$, and

$$\xi_1(n)W_i<\tilde{W}_{i,n}<\xi_2(n)W_i,\quad \xi_3(n)V_i<\tilde{V}_{i,n}<\xi_4(n)V_i,\quad \xi_5(n)u<u_n<\xi_6(n)u,$$

which imply

$$\xi_1(n)W_i+\xi_5(n)u<\tilde{W}_{i,n}+u_n<\xi_2(n)W_i+\xi_6(n)u,$$

and

$$\xi_3(n)V_i+\xi_5(n)u<\tilde{V}_{i,n}+u_n<\xi_4(n)V_i+\xi_5(n)u.$$

Then we have

$$\frac{\min\{\xi_1(n),\xi_5(n)\}(W_i+u)}{\max\{\xi_4(n),\xi_6(n)\}(V_i+u)}<\frac{\xi_1(n)W_i+\xi_5(n)u}{\xi_4(n)V_i+\xi_6(n)u}<\frac{\tilde{W}_{i,n}+u_n}{\tilde{V}_{i,n}+u_n}<$$

$$\frac{\xi_2(n)W_i+\xi_6(n)u}{\xi_3(n)V_i+\xi_5(n)u}<\frac{\max\{\xi_2(n),\xi_6(n)\}(W_i+u)}{\min\{\xi_3(n),\xi_5(n)\}(V_i+u)},$$

and

$$\frac{\min\{\xi_3(n),\xi_5(n)\}(V_i+u)}{\max\{\xi_2(n),\xi_6(n)\}(W_i+u)}<\frac{\tilde{V}_{i,n}+u_n}{\tilde{W}_{i,n}+u_n}<\frac{\max\{\xi_4(n),\xi_6(n)\}(V_i+u)}{\min\{\xi_1(n),\xi_5(n)\}(W_i+u)}.$$

Notice that

$$\Pr\left\{\max_{i\le n}\frac{\min\{\xi_1(n),\xi_5(n)\}(W_i+u)}{\max\{\xi_4(n),\xi_6(n)\}(V_i+u)}\le nx-1,\ \max_{i\le n}\frac{\min\{\xi_3(n),\xi_5(n)\}(V_i+u)}{\max\{\xi_2(n),\xi_6(n)\}(W_i+u)}\le ny-1\right\}$$

$$\le\Pr\left\{\max_{i\le n}\frac{\tilde{W}_{i,n}+u_n}{\tilde{V}_{i,n}+u_n}\le nx-1,\ \max_{i\le n}\frac{\tilde{V}_{i,n}+u_n}{\tilde{W}_{i,n}+u_n}\le ny-1\right\}$$

$$\le\Pr\left\{\max_{i\le n}\frac{\max\{\xi_2(n),\xi_6(n)\}(W_i+u)}{\min\{\xi_3(n),\xi_5(n)\}(V_i+u)}\le nx-1,\ \max_{i\le n}\frac{\max\{\xi_4(n),\xi_6(n)\}(V_i+u)}{\min\{\xi_1(n),\xi_5(n)\}(W_i+u)}\le ny-1\right\}$$

Since $\left(\xi_j(n)+\xi_k(n)\right)/\left(\xi_l(n)+\xi_m(n)\right)$ converges to 1 almost surely for all j, k, l, m, and $\max_{i\le n}\left\{(W_i+u)/(V_i+u)\right\}$ and $\max_{i\le n}\left\{(V_i+u)/(W_i+u)\right\}$ are asymptotically independent, so by Slutsky's Theorem and Theorem 2.3, we have both the first probability and the last probability in the above inequalities converging to $e^{-\left(1-e^{-1/u}\right)/x-\left(1-e^{-1/u}\right)/y}$ as $n \to \infty$, hence the middle one converges to the same joint distribution function $e^{-\left(1-e^{-1/u}\right)/x-\left(1-e^{-1/u}\right)/y}$. The proof of the asymptotic distribution of $n\tilde{q}_n$ is then similar to the proof of the asymptotic distribution $n\tilde{q}_n$ in Theorem 2.3.

REFERENCES

Bollerslev, Tim. 1986. "Generalized Autoregressive Conditional Heteroskedasticity." Journal of Econometrics 31: 307-327. http://dx.doi.org/10.1016/0304-4076(86)90063-1

Bouye, Eric. 2002. "Multivariate Extremes at Work for Portfolio Risk Measurement." Working paper, Financial Econometrics Research Centre, CUBS, London & HSBC Asset Management Europe (SA), Paris.

Buhl, Christian, Christian Reich, and Patrick Wegmann. 2002. "Extremal Dependence Between Return Risk and Liquidity Risk: An Analysis for the Swiss Market." Manuscript, University of Basel.

Chamu Morales, Francisco. 2005. "Estimation of Max-Stable Processes Using Monte Carlo Methods with Applications to Financial Risk Assessment." Ph.D. Dissertation, Department of Statistics, University of North Carolina.

Cochrane, John H. 2005. Asset Pricing (Revised edition). Princeton University Press.

Coles, Stuart G. and Jonathan A. Tawn. 1994. "Statistical Methods for Multivariate Extremes: An Application to Structural Design." Applied Statistics 43: 1-48. http://dx.doi.org/10.2307/2986112

de Haan, Laurens and Sidney I. Resnick. 1977. "Limit Theory for Multivariate Sample Extremes." Z.Wahrscheinlichkeitstheorie verw. Gebiete 40: 317-337. http://dx.doi.org/10.1007/BF00533086

Dias, Alexandra and Paul Embrechts. 2003. "Dynamic Copula Models for Multivariate High-Frequency Data in Finance." Web reference, ETHZ.

Diebold, Francis X., Til Schuermann, and John D. Stroughair. 1998. "Pitfalls and Opportunities in the Use of Extreme Value Theory in Risk Management." Journal of Risk Finance 1: 30-36. http://dx.doi.org/10.1108/eb043443

Embrechts, Paul, Claudia Klüppelberg, and Thomas Mikosch. 1999, Modelling Extremal Events for Insurance and Finance, Springer, Berlin.

Engle, Robert. 2002. "Dynamic Conditional Correlation—A Simple Class of Multivariate GARCH Models." Journal of Business and Economic Statistics 20: 339-350. http://dx.doi.org/10.1198/073500102288618487

Heffernan, Janet E. and Jonathan A. Tawn. 2004. "A Conditional Approach for Multivariate Extreme Values." Journal of Royal Statistical Society B: 497-530. http://dx.doi.org/10.1111/j.1467-9868.2004.02050.x

Ledford, Anthony W. and Jonathan A. Tawn. 1996. "Statistics for Near Independence in Multivariate Extreme Values." Biometrika 83: 169-187. http://dx.doi.org/10.1093/biomet/83.1.169

Ledford, Anthony W. and Jonathan A. Tawn. 2003. "Diagnostics for Dependence within Time Series Extremes." Journal of Royal Statistical Society B: 521-543. http://dx.doi.org/10.1111/1467-9868.00400

Lee, Sang-Won and Bruce E. Hansen. 1994. "Asymptotic Theory for The GARCH(1,1) Quasi-Maximum Likelihood Estimator." Econometrics Theory 10: 29-52. http://dx.doi.org/10.1017/S0266466600008215

Longin, François and Bruno Solnik. 2001. "Extreme Correlation of International Equity Markets." Journal of Finance 56: 649-676. http://dx.doi.org/10.1111/0022-1082.00340

Mashal, Roy and Assaf Zeevi. 2002. "Beyond Correlation: Extreme Co-Movements between Financial Assets." SSRN: http: //ssrn.com/abstract=317122, DOI: 10.2139/ssrn.317122. http://dx.doi.org/10.2139/ssrn.317122

McNeil, Alexander J. and Rüdiger Frey. 2000. "Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: An Extreme Value Approach," Journal of Empirical Finance 7: 271-300. http://dx.doi.org/10.1016/S0927-5398(00)00012-8

Mikosch, Thomas. 2003. "Modeling Dependence and Tails of Financial Time Series." Extreme Value in Finance, Telecommunications and the Environment. Chapman and Hall/CRC.

Neftci, Salih N. 2000. "Value at Risk Calculations, Extreme Events, and Tail Estimation." Journal of Derivatives 5: 23-37. http://dx.doi.org/10.3905/jod.2000.319126

Nelson, Daniel B. 1991. "Conditional Heteroskedasticity in Asset Returns: A New Approach." Econometrica 59: 347-370. http://dx.doi.org/10.2307/2938260

Pickands, James. 1975. "Statistical Inference Using Extreme Order Statistics." Annals of Statistics 3: 119-131. http://dx.doi.org/10.1214/aos/1176343003

Poon, Ser-Huang, Michael Rockinger, and Jonathan A. Tawn. 2004. "Extreme Value Dependence in Financial Markets: Diagnostics, Models, and Financial Implications." Review of Financial Studies 17: 581-610. http://dx.doi.org/10.1093/rfs/hhg058

Schalter, Martin and Jonathan A. Tawn. 2003。 "A Dependence Measure for Multivariate and Spatial Extreme Values: Properties and Inference." Biometrika 90: 139-156. http://dx.doi.org/10.1093/biomet/90.1.139

Shinki, Kazuhiko and Zhengjun Zhang. 2008. "Asymptotic Theory for GARCH Models with Stationary and Ergodic Rescaled Errors. " Technical report, Department of Statistics, the University of Wisconsin-Madison.

Smith, Richard L. 2003. "Statistics of Extremes with Applications in the Environment Insurance and Finance." Finkenstadt, B., Rootzén, H. (Eds.), Extreme Value in Finance Telecommunications and the Environment 1-78.

Smith, Richard L. and Ishay Weissman. 1996. "Characterization and Estimation of the Multivariate Extremal Index." Manuscript, University of Carolina.

Starica, Catalin. 1999. "Multivariate Extremes for Models with Constant Conditional Correlations." Embrechts, P. (Ed.), Extremes and Integrated Risk Management. Risk Books, London.

Starica, Catalin. 2000. "Multivariate Extremes for Models with Constant Conditional Correlations." Journal of Empirical Finance 6: 515-553. http://dx.doi.org/10.1016/S0927-5398(99)00018-3

Taleb, Nassim N. 2007. The Black Swan: The Impact of the Highly Improbable (1), Random House.

Tawn, Jonathan. A. 1990. "Modelling Multivariate Extreme Value Distribution." Biometrika 77: 245-253. http://dx.doi.org/10.1093/biomet/77.2.245

Tsay, Ruey S. 2005. Analysis of Financial Time Series (2nd). Wiley-Interscience. http://dx.doi.org/10.1002/0471746193

Zhang, Zhengjun. 2005. "A New Class of Tail-Dependent Time Series Models and Its Applications in Financial Time Series." Advances in Econometrics 20B: 323-358.

Zhang, Zhengjun. 2008. "Quotient Correlation: A Sample Based Alternative to Pearson's Correlation." Annals of Statistics 36: 1007-1030. http://dx.doi.org/10.1214/009053607000000866

Zhang, Zhengjun and Richard L. Smith. 2004. "The Behavior of Multivariate Maxima of Moving Maxima Processes." Journal of Applied Probability 41: 1113-1123. http://dx.doi.org/10.1239/jap/1101840556

Zhang, Zhengjun and Richard L. Smith. 2010. "On the Estimation and Application of Max-Stable Processes." Journal of Statistical Planning and Inference, 140: 1135-1153. http://dx.doi.org/10.1016/j.jspi.2009.10.014

Zheng, Shurong, Ning-Zhong Shi, and Zhengjun Zhang. 2012. “Generalized Measures of Correlation for Asymmetry, Nonlinearity and Beyond.” Journal of American Statistical Association, 107: 1239-1252. http://dx.doi.org/10.1080/01621459.2012.710509

Received on 14-11-2012 Accepted on 29-11-2012 Published on 08-12-2012

DOI: http://dx.doi.org/10.6000/1929-7092.2012.01.6