Stochastic National Income[#]

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Abstract: How can national income be adjusted to indicate welfare improvement if the future is uncertain? The present paper extends the definition of national income to stochastic settings on the basis of discounted utilitarian welfare function. Real interest rate of consumption is redefined so that real national income can be interpreted as the *expected* present value of real interest on future national consumption. A stochastic one-good model is used to illustrate the application of the stochastic real national income. It turns out that under uncertainty real national income may be decreasing even though captial stock is constant or increasing over time.

Keywords: Stochastic income, comprehensive national accounting.

1. INTRODUCTION

In order to response to the public concerns on the environment, national accountants are seeking a proper indicator of welfare improvement. Several indicators have been proposed such as genuine savings by Hamilton (1994), national income by Sefton and Weale (2006), and comprehensive wealth by Dasgupta (2009). In the present paper, I will focus on national income and explore whether and how national income in a stochastic setting can be used to indicate welfare improvement. I would like to start from the definition of income proposed by Hicks (1946), p.~172), who defines "...a man's income as the maximum value which he can consume during a week, and still *expect* to be as well off at the end of the week as he was at the beginning." By such a definition (Hicksian income), Hicks(1946) intends to make income an indicator to give people an amount "which they can consume without impoverishing themselves." The definition specifies three key elements:

- Income corresponds to a maximum value of consumption in a period satisfying certain assumptions.
- There is something that should be "as well off" if the income is consumed in the period.
- The level "as well off" is an expectation and hence income itself is an expectation.

In the literature of comprehensive national accounting theory, there is no doubt on the first element, i.e. income is always interpreted as a maximum value of consumption satisfying certain conditions. However, there are several interpretations for the second element. What should be "as well off?"

Following the tradition of Fisher (1906) and Lindahl (1933, Section II), income is defined as interest on wealth, where wealth is the present value of future consumption. The definition suggests keeping wealth "as well off." If the interest rate is constant all the time, then constant wealth provides a constant income flow. Otherwise, the constant wealth can not ensure a constant income flow.

In the spirits of " Income No. 3" offered by Hicks (1946), income can be represented by the "*stationary equivalent* of future consumption" (Weitzman, 1976), p.~160). This concept of income intends to keep *consumption* "as well off" and is related to the constant level of consumption with the same present value as the actual future stream of consumption in an economy where well-being depends on a single consumption good. Unfortunately, as pointed out by Asheim (1997), Sefton and Weale (2006, Section 3.1.2) and Asheim and Wei (2009, Appendix B), such *wealth equivalent income* does not satisfy certain properties and is difficult to generalize to the empirically relevant case of multiple consumption goods.

Another attempt (Pemberton and Ulph 2001, Sefton and Weale 2006) is to associate " as well off" with the level of dynamic welfare. Following the view, Sefton and Weale (2006) define national income as a weighted present value of future national consumption, which is further interpreted as the present value of real interest on future national consumption by Asheim and Wei (2009).

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The third key element of the Hicksian income mentioned at the beginning is ignored in the literature for a long time. When the theory is applied to practice, the uncertainty is an unavoidable problem. Then it is better to interpret the level "as well off" as an expectation and so for the concept of income.

By applying the maximum principle of control theory, Weitzman (2003, Chapter 7) introduces a stochastic process to the capital stock in a one-good stochastic model and a concept of income in a stochastic setting is defined as the return on expected wealth (Weitzman 2003, p.~321), where wealth can be understood as the present value of future consumption discounted by constant rate.

In the present paper I extend the definition of national income by Sefton and Weale (2006) to stochastic settings in a multi-good economy. As shown in the citations from Hicks (1946), The Hicksian income is estimated for the purpose of prudent behavior. It means the income is estimated to make the decision on how much to consume for the current period. Hence, the income has to be calculated at the beginning of the period before the current decision is made, which can be called *ex ante* income. Obviously *ex post* income, which is calculated after the current decision is made, is irrelevant to the decision-making process for the current period. On the contrary, the current decision may have effects on the *ex post* income. Then, the present paper focuses on *ex ante* income. The results of the paper reminds us that national income estimated in the deterministic manner may no longer serve as an indicator for prudent behavior under uncertainty.

The remainder of the paper is organized as follows. Next section illustrates the theory of national income in the deterministic case. Then stochastic processes are introduced in the analysis to define stochastic national income in Section 3. In Section 4, a stochastic one-good model is used to illustrate the findings in the paper. The final section concludes the paper. In the Appendix I provide the proves of propositions in the paper.

2. DETERMINISTIC NATIONAL INCOME

As mentioned above, the concept of income is a maximum of consumption level such that the dynamic welfare keeps at the same level at the end of each time. Then the first step is to know what the dynamic welfare is. In the literature, the dynamic welfare is generally related to utility and then consumption, namely all factors that have effects on utility are defined to be included in the concept of "consumption." Hence, any instantaneous change of dynamic welfare can be associated with the consumption change. It is proved that the present value of future consumption change measures welfare improvement by properly defined prices of consumption (ref. Samuelson, 1961; Sefton and Weale, 2006; Asheim and Wei, 2009). By interpreting the present value of future consumption changes as national savings and adding to current consumption (measured in the same numeraire), we get the concept of national income. Furthermore, real national income can be defined as the present value of real interest on future consumption (ref. Asheim and Wei, 2009).

For a given unidimensional utility flow $\{U(s)\}_{s=0}^{\infty}$ over time, dynamic welfare at any time $t \geq 0$ is defined by the discounted utilitarian,

$$W(t) = \int_t^{\infty} e^{-\rho(s-t)} U(s)\, ds,$$

where ρ is a given constant utility discount rate. Then all the future utility has effect on the present dynamic welfare. By Leibniz's Formula, the instantaneous change of welfare is represented by

$$\frac{dW(t)}{dt} = -U(t) + \rho \int_t^{\infty} e^{-\rho(s-t)} U(s)\, ds.$$

Furthermore, by assuming the utility flow is a smooth curve over time, i.e. the first-order derivative of utility w.r.t. time exists almost everywhere, and the transversality condition holds: $\lim_{s\to\infty} e^{-\rho(s-t)} U(s) = 0,$ which implies the infinite future utility means nothing for current time, then we obtain by integrating by parts

$$\frac{dW(t)}{dt} = \int_t^{\infty} e^{-\rho(s-t)} \dot{U}(s)\, ds, \tag{1}$$

which shows that instantaneous change of welfare over time depends on the present value of future utility change (ref. Asheim, 2007).

Now assume the utility is a time-invariant, concave and non-decreasing function u with continuous second derivatives w.r.t. an n-dimensional vector of consumption $\mathbf{C}$. Suppose the utility flow is given by

$$U(s) = u(\mathbf{C}(s)), \tag{2}$$

for all $s \geq 0$, where $\mathbf{C}(s)$ is the consumption at time $s \geq 0$. Define present value consumer prices, $\{\mathbf{p}_c(s)\}_{s=0}^{\infty}$ for all $s \geq 0$, satisfying,

$$\mathbf{p}_c(s) = e^{-\rho s}\nabla u(\mathbf{C}(s)),$$

where $\nabla u(\mathbf{C}(s))$ represents the vector of marginal utility w.r.t. consumption at the time s, i.e.

$$\nabla u(\mathbf{C}(s)) = \left(\frac{\partial u}{\partial c_1}(\mathbf{C}(s)), \frac{\partial u}{\partial c_2}(\mathbf{C}(s)), \ldots, \frac{\partial u}{\partial c_n}(\mathbf{C}(s))\right).$$

In addition we still assume the transversality condition holds: $\lim_{s\to\infty} e^{-\rho s}u(\mathbf{C}(s)) = 0$. By applying (2) to (1), we achieve that for a given smooth consumption flow over time[1],

$$\frac{dW(t)}{dt} = \int_t^\infty e^{-\rho(s-t)}\nabla u(\mathbf{C}(s))\dot{\mathbf{C}}(s)\,ds = e^{\rho t}\int_t^\infty \mathbf{p}_c(s)\dot{\mathbf{C}}(s)\,ds,$$

which shows that instantaneous change of welfare is represented by the present value of future consumption change. As shown in the previous literature (eg. Samuelson, 1961; Sefton and Weale, 2006; Asheim, 2007), $\int_t^\infty \mathbf{p}_c(s)\dot{\mathbf{C}}(s)ds$ can be interpreted as national savings. Then, if national income is to serve as a guide for prudent behavior such that the dynamic welfare improves if and only if national consumption is smaller than national income, we obtain a definition of national income as the sum of current consumption value plus the present value of future consumption change,

$$y(t) = \mathbf{p}_c(t)\mathbf{C}(t) + \int_t^\infty \mathbf{p}_c(s)\dot{\mathbf{C}}(s)\,ds. \tag{3}$$

Just by integrating by parts, we directly derive another expression of national income from (3),

$$y(t) = \int_t^\infty -\dot{\mathbf{p}}_c(s)\mathbf{C}(s)\,ds, \tag{4}$$

since $u(\mathbf{C})$ has continuous second derivatives w.r.t. consumption.

A Divisia consumer price index $\{\pi(s)\}_{s=0}^\infty$ can be defined satisfying $\pi(0) = 1$ and

$$\frac{\dot{\pi}(s)}{\pi(s)} = \frac{\dot{\mathbf{p}}_c(s)\mathbf{C}(s)}{\mathbf{p}_c(s)\mathbf{C}(s)} \tag{5}$$

for all $s \geq 0$. Define the path of *real* consumption interest rates $\{R(s)\}_{s=0}^\infty$ by

$$R(s) = -\frac{\dot{\pi}(s)}{\pi(s)} \tag{6}$$

for all $s \geq 0$. Then for all $s \geq 0$,

$$\pi(s) = \exp\left(\int_0^s -R(v)\,dv\right). \tag{7}$$

The real consumption price flow $\{\mathbf{P}_c(s)\}_{s=0}^\infty$ can be defined by

$$\mathbf{P}_c(s) = \frac{\mathbf{p}_c(s)}{\pi(s)} \tag{8}$$

for all $s \geq 0$. Since for all $s \geq 0$,

$$-\dot{\mathbf{p}}_c(s)\mathbf{C}(s) = -\frac{\dot{\pi}(s)}{\pi(s)}\mathbf{p}_c(s)\mathbf{C}(s) \text{ by (5)}$$

$$= R(s)\mathbf{p}_c(s)\mathbf{C}(s) \text{ by (6)}$$

$$= R(s)\pi(s)\mathbf{P}_c(s)\mathbf{C}(s) \text{ by (8)},$$

Then by (4), real national income is associated with *the sum of present value of real interest on future national consumption* as stated in the following definition (Sefton and Weale 2006, Asheim and Wei 2009).

Definition 1 Real national income at time t is defined as

$$Y(t) = \int_t^\infty \frac{\pi(s)}{\pi(t)}R(s)\mathbf{P}_c(s)\mathbf{C}(s)\,ds. \tag{9}$$

where $\mathbf{C}(s)$ is the consumption at time $s \geq 0$.

3. NATIONAL INCOME IN STOCHASTIC SETTINGS

A natural question now is what the definition of national income will be if the economy is not deterministic. In this section I introduce Brownian motions (or Wiener processes) into the model and derive expressions of national income for the stochastic case. The analysis in the section involves stochastic integrals. All the results can be expressed by the common Itô integral, which is better for the real calculation. However, in order to interpret the results more intuitively, another type of stochastic integral, so-called Stratonovich integral, is used to express the results following Weitzman (2003). The latter type of integral exhibits the useful property of chain rule, which enables us to derive expressions in the stochastic case similar to the deterministic case.

[1]In the paper, the product of any two vectors means the inner-product of the two vectors. eg. if $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, then $\mathbf{pc} = \sum_{i=1}^n p_i c_i$. Then we always have the Commutative Law, $\mathbf{pc} = \mathbf{cp}$.

3.1. Welfare and National Income

Suppose a unidimensional utility flow $\{U(s)\}_{s=0}^{\infty}$ with given initial value $U(0)$ is an Itô process, i.e. the utility has the form

$$dU(s) = a(s,\omega)\,ds + b(s,\omega)\,dB(s)$$

for all $s \geq 0$, where ω is random variables, B is a unidimensional Brownian motion, and $a(s) = E[dU(s)]/ds$, the expected instantaneous change of utility at time s. Dynamic welfare at any time $t \geq 0$ is defined by the expected discounted utilitarian,

$$W(t) = E^t\left[\int_t^{\infty} e^{-\rho(s-t)} U(s)\,ds\right], \tag{10}$$

where ρ is a constant utility discount rate. If instantaneous welfare improvement is represented by the expected instantaneous change of welfare at any time $t \geq 0$,

$$AW(t) = \lim_{h \to 0^+} \frac{E^t[W(t+h)] - W(t)}{h}, \tag{11}$$

which is formally called the infinitesimal generator of $W(t)$ in mathematics (Øksendal, 2005, Section 7.3), then we have

Proposition 1 If utility in the infinite future is assumed to mean nothing for current economy, i.e. $\lim_{s\to\infty} e^{-\rho(s-t)} E^t[U(s)] = 0$ for any given point in time $t \geq 0$, then instantaneous welfare improvement, i.e. the expected instantaneous change of welfare at any time $t \geq 0$, $AW(t)$, is represented by the expected discounted value of future utility change,

$$AW(t) = E^t\left[\int_t^{\infty} e^{-\rho(s-t)}\,dU(s)\right]. \tag{12}$$

Now let the utility at any future time defined as a time-invariant, concave and non-decreasing function u with continuous second derivatives w.r.t. an n-dimensional vector of consumption $\mathbf{C}$, i.e. $U(t) = u(\mathbf{C}(t))$ for all $t \geq 0$, where consumption is an Itô process with given initial value $\mathbf{C}(0)$. i.e. the vector of consumption has the form

$$d\mathbf{C}(t) = \boldsymbol{\alpha}(t,\omega)\,dt + \boldsymbol{\beta}(t,\omega)\,dB_t$$

for all time $t \geq 0$, where $\boldsymbol{\alpha}$ is an n-dimensional vector, ω random variables, $\boldsymbol{\beta}$ an $n \times m$ dimensional vector, and B is an m-dimensional Brownian motion. Then the utility flow $\{U(t)\}_{t=0}^{\infty}$ is also an Itô process. Still let $\nabla u(C(t))$ denote the vector of marginal utility w.r.t. consumption at any time $t \geq 0$, and the vector of the present value prices of consumption is defined satisfying

$$\mathbf{p}_c(s) = e^{-\rho(s-t)} \nabla u(\mathbf{C}(s)) \tag{13}$$

for all $s \geq t$. Then Proposition 1 can be applied to obtain one corollary:

Corollary 1 If the utility due to consumption in the infinite future is assumed to mean nothing for the current economy, i.e. $\lim_{s\to\infty} E^t\left[e^{-\rho(s-t)} u(\mathbf{C}(s))\right] = 0$ for any given point in time $t \geq 0$, then the instantaneous welfare improvement is indicated by the expected present value of future consumption change,

$$AW(t) = E^t\left[\int_t^{\infty} \mathrm{p}_c(s) \circ d\mathrm{C}(s)\right], \tag{14}$$

where $\circ d$ indicates Stratonovich integral.

Since the r.h.s. of expression (14) is a Stratonovich integral, the consumption prices used for evaluation can be called the *Stratonovich-like prices of consumption*.

Corollary 1 gives a welfare foundation for interpreting $E^t\left[\int_t^{\infty} \mathbf{p}_c(s) \circ d\mathbf{C}(s)\right]$ as national savings. Hence, if national income is supposed to be a guide for prudent behavior such that the dynamic welfare improves if and only if national consumption is smaller than national income, then *national income* equals an expectation, $\mathbf{p}_c(t)\mathbf{C}(t) + E^t\left[\int_t^{\infty} \mathbf{p}_c(s) \circ d\mathbf{C}(s)\right]$. By using the chain rule of Stratonovich integral on $\mathbf{p}_c(s)\mathbf{C}(s)$, we have

$$\lim_{s\to\infty} \mathbf{p}_c(s)\mathbf{C}(s) - \mathbf{p}_c(t)\mathbf{C}(t) = \int_t^{\infty} \mathbf{p}_c(s) \circ d\mathbf{C}(s) + \int_t^{\infty} \mathbf{C}(s) \circ d\mathbf{p}_c(s).$$

If the consumption in the infinite future means nothing for the current economy, i.e.

$$\lim_{s\to\infty} E^t[\mathbf{p}_c(s)\mathbf{C}(s)] = 0,$$

Then we obtain

$$\int_t^{\infty} \mathbf{C}(s) \circ (-d\mathbf{p}_c(s)) = \mathbf{p}_c(t)\mathbf{C}(t) + \int_t^{\infty} \mathbf{p}_c(s) \circ d\mathbf{C}(s).$$

By taking the expectation on both sides, we find another expression for *national income*,

$$\underbrace{E^t\left[\int_t^{\infty} \mathbf{C}(s) \circ (-d\mathbf{p}_c(s))\right]}_{\text{national income}} = \underbrace{\mathbf{p}_c(t)\,\mathbf{C}(t)}_{\text{current consumption}} + \underbrace{E^t\left[\int_t^{\infty} \mathbf{p}_c(s) \circ d\mathbf{C}(s)\right]}_{\text{national savings}}. \tag{15}$$

The equation is the main result of the subsection. Under uncertainty, national income is an expectation. Since current consumption is known, then the uncertainty comes from the national savings.

A *One-Dimensional Example* for *National Savings*

If the consumption is one-dimensional and follows the form,

$$dC(t) = \alpha(t)dt + \beta(t)d\mathbf{B}_t,$$

where α and β are given paths of constants over time, then by the relations between Itô and Stratonovich integrals, national savings can be expressed by the Itô integral,

$$\underbrace{E^t\left[\int_t^\infty p_c(s)\circ dC(s)\right]}_{\text{national savings}} = E^t\left[\int_t^\infty p_c(s)\alpha(t)ds\right] + \frac{1}{2}E^t\left[\int_t^\infty \frac{dp_c(s)}{dc}(C(s))(\beta(t))^2 ds\right],$$

where the first term on the r.h.s. can be calculated on the basis of the expected instantaneous change of future consumption and their expected prices. The second term is an adjustment term, which depends on the correlations between future consumption and their prices. For instance, if the demand curve is given by a downward-sloping linear function, $p_c(s) = e^{-\rho(s-t)}\left[\overline{p}_c - b_c C(s)\right]$, where $b_c > 0$, then the adjustment term equals

$$\frac{1}{2}E^t\left[\int_t^\infty e^{-\rho(s-t)}(-b_c)(\beta(s))^2 ds\right] = -\frac{1}{2}b_c E^t\left[\int_t^\infty e^{-\rho(s-t)}(\beta(s))^2 ds\right] \le 0,$$

which implies national savings should be adjusted downwards and so should national income.

3.2. National Income and NNP

So far the definition of the stochastic national income by (15) has a sound welfare interpretation. By adding more assumptions, we can show that stochastic national income coincides with net national product (NNP). NNP is defined as the sum of current consumption value plus the expected value of current capital change evaluated at a Stratonovich-like capital price, the latter can also be called genuine savings (the terminology introduced by Hamilton, 1994). Notice that The NNP is redefined here such that it is evaluated after the investment is decided and just before the stochastic factor on capital stock is revealed at current time.

Suppose the state of the economy is represented by an m-dimensional vector of capital stock $\mathbf{K}$, which is known at current time and used to produce more goods. Besides consumption, commodities produced can be invested to change the capital stock flow over time. The consumption - net investment pair $(\mathbf{C},\mathbf{I})$ is called attainable if $(\mathbf{C}(s),\mathbf{I}(s)) \in S(\mathbf{K}(s))$ for time $s \ge 0$, where $S(\mathbf{K})$ is the production probabilities sets given capital $\mathbf{K}$. If the level of capital $\mathbf{K}$ at any two points in time is the same, then the allocated consumption and net investment are the same for these two points. Thus, both consumption and net investment at each point in time are functions of current capital alone, i.e. $\mathbf{C}(s) = \mathbf{C}(\mathbf{K}(s))$ and $\mathbf{I}(s) = \mathbf{I}(\mathbf{K}(s))$ hold for any time $s \ge 0$. Such an arrangement is called a *resource allocation mechanism* (RAM) (as introduced Dasgupta and Mäler, 2000; Dasgupta, 2001; Arrow, *et al*., 2003).

Suppose the capital change at each time is affected by two factors: the net investment and a random variable with zero expectation. Then let a RAM decided at the beginning of current time 0, the net investment depends on capital alone and the expected instantaneous capital change can be expressed by a function of the net investments alone, i.e. $E[d\mathbf{K}(s)]/ds = \boldsymbol{\mu}_K(\mathbf{I}(\mathbf{K}(s)))$ for all $s \ge 0$. Thus, let the capital stock flow $\{\mathbf{K}(s)\}_{s=0}^\infty$ with given initial value $\mathbf{K}(0)$ represented by an autonomous Itô diffusion of the form[2]

$$d\mathbf{K}(s) = \boldsymbol{\mu}_K(\mathbf{I}(\mathbf{K}(s)))ds + \boldsymbol{\sigma}(\mathbf{K}(s))dB(s), \qquad (16)$$

for all $s \ge 0$, where B is an m-dimensional Brownian motion over time. The Brownian motion represents the stochastic factor with zero expectation that has effects on capital stock. Suppose the stochastic differential equation has a unique solution given the initial capital. Hence, the capital stock at any point in time $s \ge 0$ is determined by the initial capital stock and the Brownian motion up to the point in time s. i.e.

$$\mathbf{K}(s) = f\left(\mathbf{K}(0),\{B(v)\}_{v=0}^s\right) = f\left(\mathbf{K}(t),\{B(v)\}_{v=t}^s\right)$$

where the second equation holds for any $0 \le t \le s$.

Then given the RAM in the economy, the consumption flow $\{\mathbf{C}(s)\}_{s=t}^\infty$ is determined by the initial given capital stock $\mathbf{K}_t$ and the Brownian motion. If further the utility is still assumed to be a function of consumption alone, i.e. $U(t) = u(\mathbf{C}(t))$ for all time $t \ge 0$, then the dynamic welfare in the economy is uniquely determined by the initial condition of the capital stock since

[2]An intuitive interpretation for the stochastic capital is provided by Weitzman (2003, Chapter 7).

$$W(t) = E^t\left[\int_t^\infty e^{-\rho(s-t)}U(s)\,ds\right] = E^t\left[\int_t^\infty e^{-\rho(s-t)}\tilde{u}\left(\mathbf{K}(t),\{B(s)\}_{s=t}^{s}\right)ds\right], \quad (17)$$

where

$$\tilde{u}\left(\mathbf{K}(t),\{B(s)\}_{v=t}^{s}\right) = u\left(\mathbf{C}\left(f\left(\mathbf{K}(t),\{B(v)\}_{v=t}^{s}\right)\right)\right).$$

Thus, the dynamic welfare expressed by (17) can be rewritten as a function of the current capital,

$$W(t) = \tilde{w}(\mathbf{K}(t)),$$

which is assumed to be twice continuously differentiable w.r.t. capital $\mathbf{K}$. In addition, the vector of prices of net investment at time t is defined to equal the vector of marginal welfare w.r.t. current capital $\mathbf{K}$, i.e. $\mathrm{p}_k(t) = \nabla\tilde{w}(\mathbf{K}(t))$. By the chain rule of Stratonovich integral, the instantaneous change of welfare can be expressed by

$$dW(t) = \nabla\tilde{w}(\mathbf{K}(t))\circ d\mathbf{K}(t) = \mathbf{p}_k(t)\circ d\mathbf{K}(t), \quad (18)$$

which shows that instantaneous change of welfare is represented by the present value of current capital change evaluated at the Stratonovich-like capital prices. Then national income can be shown to equal NNP, the sum of current consumption value plus the genuine savings.

Proposition 2 By specifying a resource allocation mechanism (RAM), dynamic welfare is defined as a function with respect to the initial capital alone, which is assumed to be twice continuously differentiable. Then, national income coincides with the NNP[3], i.e.

$$E^t\left[\int_t^\infty \mathbf{C}(s)\circ(-d\mathbf{p}_c(s))\right] = \mathbf{p}_c(t)\mathbf{C}(t) + \frac{E^t\left[\mathbf{p}_k(t)\circ d\mathbf{K}(t)\right]}{dt}, \quad (19)$$

where the second term on the r.h.s. is the expected value of net investments ex ante, which is evaluated just before the uncertainty on current capital is revealed.

A One-Dimensional Example for NNP

Assume the capital is one-dimensional. By the relations between Itô and Stratonovich integrals, national income can be expressed as NNP,

$$y(t) = p_c(t)C(t) + p_k(t)I(t) + \frac{1}{2}\frac{d\mathbf{p}_k}{d\mathbf{k}}(\mathbf{K}(t))\sigma^2(t). \quad (20)$$

If the capital and its price is deterministic, i.e. $\sigma(t)=0$ for all $t\geq 0$, then the last term on the r.h.s. of (20) disappears. Hence, if the capital is a stochastic process, national income ex ante calculated should be adjusted to include the value of all types of capital changes, including the uncertain part. The adjustment term depends on the correlations between future capital and its prices. For instance, if for any given capital level, its price is given by a linear function, $p_k(s) = \bar{p}_k - b_k K(s)$, where $b_k > 0$, then the adjustment term equals

$$\frac{1}{2}\frac{dp_k}{dk}(K(t))\sigma^2(t) = -\frac{1}{2}b_k\sigma^2(t) \leq 0,$$

which implies the national savings (and also the national income) should be adjusted downwards to precaution the uncertainty of the capital.

3.3. Real National Income

In this subsection, national income in real terms is defined by introducing the Divisia consumption price index and the redefined real interest rate.

Define a Divisia price index $\{\pi(s)\}_{s=0}^{\infty}$ satisfying initial value $\pi(0)=1$ and

$$\frac{d\pi(s)}{\pi(s)} = \frac{\mathbf{C}(s)\,d\mathbf{p}_c(s)}{\mathbf{C}(s)\,\mathbf{p}_c(s)} \quad (21)$$

for all $s\geq 0$. Notice that the index is always a stochastic process since future consumption are stochastic and so their prices are uncertain. The real consumption price $\mathbf{P}_c$ is defined by

$$\mathbf{P}_c(s) = \frac{\mathbf{p}_c(s)}{\pi(s)} \quad (22)$$

for all $s\geq 0$.

To derive the national income in real terms, we further define real interest rate of consumption (R) as the expected instantaneous change rate of the present value of consumption due to price change alone, i.e.

$$R(s) = \frac{1}{\mathbf{C}(s)\mathbf{p}_c(s)}\lim_{h\to 0^+}\frac{1}{h}E^s\left[\int_s^{s+h}\mathbf{C}(v)\circ(-d\mathbf{p}_c(v))\right] \quad (23)$$

[3]The formula in Weitzman (2003, Chapter 7.) can be taken as a special expression of NNP defined here.

for all $s \geq 0$. By using (23) and (22), we have national income[4]

$$E^t\left[\int_t^\infty \mathbf{C}(s)\circ(-d\mathbf{p}_c(s))\right] = E^t\left[\int_t^\infty \pi(s)R(s)\mathbf{P}_c(s)\mathbf{C}(s)ds\right] \quad (24)$$

for all time $t \geq 0$. Thus, we have

Definition 2 Real national income equals the expected present value of real interest on future national consumption,

$$Y(t) = E^t\left[\int_t^\infty \frac{\pi(s)}{\pi(t)} R(s)\mathbf{P}_c(s)\mathbf{C}(s)ds\right], \quad (25)$$

which can be split as

$$Y(t) = \mathbf{P}_c(t)\mathbf{C}_t + E^t\left[\int_t^\infty \frac{\pi(s)}{\pi(t)}\mathbf{P}_c(s)\circ d\mathbf{C}(s)\right]. \quad (26)$$

The second part of Definition 2 is obtained by substituting (22) to the r.h.s. of (15).

4. A STOCHASTIC ONE-GOOD MODEL

This section illustrates the findings of the paper by a stochastic one-good model. A Cobb-Douglas production function with two inputs, labor and capital, is adopted. The labor is normalized to be unity. To highlight the effect of the stochastic factor, the capital is supposed to be constant all the time and the uncertainty comes from the technological change. At each point in time, the value of the technological change follows a normal distribution.

The dynamic welfare is expected discount utilitarian and utility function exhibits the property of constant relative risk aversion (CRRA). The RAM is determined such that all the current production is used for consumption and not for investment at all. Under these settings, the real interest rate used for income calculation is a constant all the time. On the contrary, the capital prices are stochastic and its expectation differs from the constant real interest rate.

Assume the output from the production depends also on a normal-distributed technological variable z_t besides the constant capital k at each time t (no depreciation for the capital). Hence, production, $q(t)$ at time t is given by

$$q(t) = e^{z_t} k^\alpha$$

where the available labor ℓ is constant and normalized to one (i.e., $\ell(t) = 1$ for all t), which is ignored in the function. We can let $z_t = \sigma B_t$, a stochastic process with initial value $z_0 = 0$, where the constant $\sigma > 0$ and B_t is a Brownian motion. We also assume that $0 < \alpha < 1$.

Suppose the resource allocation mechanism (RAM) is determined as follows. Given any time $t \geq 0$, all the production is consumed and nothing is invested, $dk(t) = 0$, which is consistent with the constant capital assumption. Then consumption

$$C(s) = q(s) = e^{\sigma B_s} k^\alpha \quad (27)$$

for all $s \geq 0$.

Next assume the utility function is given by

$$U(s) = u(C(s)) = \frac{C(s)^{1-\eta}}{1-\eta} \quad (28)$$

where the curvature of the function $\eta \geq 0$ and $\eta \neq 1$ [5]. Given the discounted utilitarian welfare function (10), by letting the current consumption price $p_c(0) = 1$, we have the present value prices of consumption

$$p_c(s) = e^{-\rho s}\frac{\partial u}{\partial c}(C(s)) / \frac{\partial u}{\partial c}(C_0) = C_0^\eta e^{-\rho s} C(s)^{-\eta} \quad (29)$$

for all $s \geq 0$. Then on the basis of (27) and (29), by Itô integral, we know

$$dC(s) = C(s)\left[\frac{1}{2}\sigma^2 ds + \sigma dB(s)\right]$$

$$dp_c(s) = -\rho p_c(s)ds + C_0^\eta e^{-\rho s} d\left(C(s)^{-\eta}\right)$$

$$= -\rho p_c(s)ds + C_0^\eta e^{-\rho s}(-\eta)C(s)^{-\eta-1}dC(s) + \frac{1}{2}C_0^\eta e^{-\rho s}$$
$$(-\eta)(-\eta-1)C(s)^{-\eta-2}(dC(s))^2$$

$$= p_c(s)\left[\left(\frac{1}{2}\eta^2\sigma^2 - \rho\right)ds - \eta\sigma dB(s)\right]$$

and

[4] Roughly speaking, the real interest rate of consumption can be thought of as $R(s) = \frac{E^s[C(s)\circ(-dp_c(s))]/ds}{C(s)p_c(s)}$. Then it can be rewritten as $E^s[C(s)\circ(-dp_c(s))] = R(s)C(s)p_c(s)ds = \pi(s)R(s)P_c(s)C(s)ds$. Hence, $E^t[C(s)\circ(-dp_c(s))] = E^t\{E^s[C(s)\circ(-dp_c(s))]\} = E^t\{\pi(s)R(s)P_c(s)C(s)ds\}$.

[5] If $\eta = 1$, then the utility function is elapsed to be the logarithmic form, $u(C(s)) = \log C(s)$.

$$C(s)\circ(-dp_c(s)) = C(s)p_c(s)\left(\rho - \frac{1}{2}(\eta^2-\eta)\sigma^2\right)$$

$$ds + C(s)p_c(s)\eta\sigma dB_s \quad (30)$$

$$p_c(s)\circ dC(s) = C(s)p_c(s)\frac{1}{2}(1-\eta)\sigma^2 ds + C(s)p_c(s)\sigma dB_s. \quad (31)$$

Hence, by definition (23), the real interest rate over time,

$$R(s) = \rho - \frac{1}{2}(\eta^2-\eta)\sigma^2 = R, \quad (32)$$

which is a constant and we suppose it is positive. Then if $\eta > 1$, the uncertainty implies smaller expectation of future real interest rate. Since this is a one-good economy, the present value prices of consumption can serve as the Divisia consumption price index, i.e. $\pi(s) = p_c(s)$ for all $s \geq 0$. Then the real prices of consumption is a constant, $P_c(s) = 1$ for all $s \geq 0$. Hence, the real interest rate includes two terms: the expected instantaneous change rate of the Divisia consumption price index π and the adjustment due to the correlations between consumption and the Divisia price index.

$$R(s) = \underbrace{\rho - \frac{1}{2}\eta^2\sigma^2}_{\tilde{R},\ \text{expected change rate of } \pi} + \underbrace{\frac{1}{2}\eta\sigma^2}_{\text{adjustment due to correlation between } C \text{ and } \pi}.$$

The adjustment term could be considerable. For example, if $\eta = 1$ and $\sigma = 0.1$, then the adjustment term $\frac{1}{2}\eta\sigma^2$ is half percent, which should not be ignored if real interest rate R is small, eg. 4 percent.

Before the calculation of real national income, we notice that

$$E^0\left[\int_0^\infty C(s)p_c(s)ds\right] = C(0)^\eta E^0\left[\int_0^\infty e^{-\rho s}C(s)^{1-\eta}ds\right] \text{ by (29)}$$

$$= k^{\alpha\eta}E^0\left[\int_0^\infty e^{-\rho s}\left(e^{\sigma B_s}k^\alpha\right)^{1-\eta}ds\right] \text{ by (27)}$$

$$= k^{\alpha\eta}k^{\alpha(1-\eta)}E^0\left[\int_0^\infty e^{-\rho s}e^{(1-\eta)\sigma B_s}ds\right]$$

$$= k^\alpha\int_0^\infty e^{-\rho s}E^0\left[e^{(1-\eta)\sigma B_s}\right]ds$$

$$= k^\alpha\int_0^\infty e^{-\rho s}e^{\frac{1}{2}(1-\eta)^2\sigma^2 s}ds$$

$$= \frac{k^\alpha}{\rho - \frac{1}{2}(1-\eta)^2\sigma^2} \quad \text{if } \rho > \frac{1}{2}(1-\eta)^2\sigma^2.$$

Hence by (26) and (31), real national income is

$$Y(0) = P_c(0)C(0) + E^0\left[\int_0^\infty \frac{p_c(s)}{p_c(0)}P_c(s)\circ dC(s)\right]$$

$$= k^\alpha + \frac{1}{2}(1-\eta)\sigma^2E^0\left[\int_0^\infty C(s)p(s)ds\right] + E^0\left[\int_0^\infty C(s)p(s)\sigma dB(s)\right]$$

$$= \underbrace{k^\alpha}_{\text{current consumption}} + \underbrace{\frac{\frac{1}{2}(1-\eta)\sigma^2k^\alpha}{\rho - \frac{1}{2}(1-\eta)^2\sigma^2}}_{\text{national savings}} \quad \text{if } \rho > \frac{1}{2}(1-\eta)^2\sigma^2$$

$$= \frac{\rho - \frac{1}{2}(\eta^2-\eta)\sigma^2}{\rho - \frac{1}{2}(1-\eta)^2\sigma^2}k^\alpha, \quad (33)$$

where national savings is negative if $\eta > 1$. Otherwise, if $\eta < 1$, national savings is positive.

We can obtain the same result by (25),

$$Y(0) = E^0\left[\int_0^\infty \frac{p_c(s)}{p_c(0)}R(s)P_c(s)C(s)ds\right]$$

$$= \left(\rho - \frac{1}{2}(\eta^2-\eta)\sigma^2\right)E^0\left[\int_0^\infty C(s)p(s)ds\right] \text{ by (32)}$$

$$= \frac{\rho - \frac{1}{2}(\eta^2-\eta)\sigma^2}{\rho - \frac{1}{2}(1-\eta)^2\sigma^2}k^\alpha \quad \textit{if } \rho > \frac{1}{2}(1-\eta)^2\sigma^2.$$

Notice that if $\rho \leq \frac{1}{2}(1-\eta)^2\sigma^2$, then the income goes to infinity over time in this case. In the following we always assume that $\rho > \frac{1}{2}(1-\eta)^2\sigma^2$.

By setting $\rho = 0.04$, Figure **1** shows the rate of real national income to the current production corresponding to various values of the curvature of utility function η and the uncertainty level of production σ. Under certainty, i.e. $\sigma = 0$, the real national income coincides with the current production irrespective of the curvature of utility function. Under uncertainty, i.e. given certain level of $\sigma \neq 0$, higher curvature of utility implies lower real national income. Notice that the real national income is just half of the current production if $\eta = 3$ and $\sigma = 0.1$. However, given the curvature of utility η, the real national income may become higher or lower along with higher level of uncertainty σ.

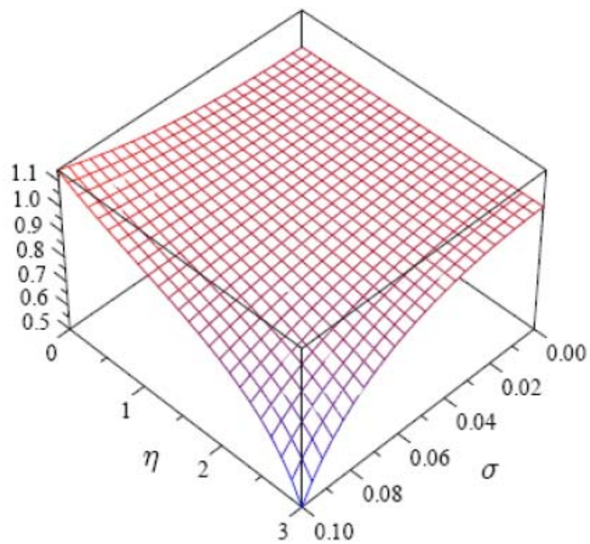

Figure 1: The change rates of real national income w.r.t. the curvature of utility function and the uncertainty level of production.

It turns out national income is estimated to differ from current production because of the uncertain future production and the curvature of utility function. For example, if we apply the linear utility function $U(s) = C(s)$, i.e. $\eta = 0$, then national income becomes

$$Y(0) = \frac{\rho}{\rho - \frac{1}{2}\sigma^2} k^{\alpha},$$

which is higher than the current production. Since utility is exactly equal to consumption all the time, this higher estimation shows the future output (and so consumption) is expected to increase, i.e. $E^0[q(t)] = E^0\left[e^{z_t} k^{\alpha}\right] = e^{\frac{1}{2}\sigma^2 t} k^{\alpha} \geq k^{\alpha}$. It can be shown that if future consumption is expected to be the same as current consumption, the linear utility function implies that national income equals current production (this can be got by assuming $dz_t = -\frac{1}{2}\sigma^2 dt + \sigma dB_t$).

If instead, we assume the utility function is given by the logarithmic form,

$$U(s) = u(C(s)) = \log C(s),$$

Then the national income becomes

$$Y(0) = k^{\alpha},$$

which is exactly the current production since the effect of curvature of utility function cancels out the expectation of higher future production.

Since the real interest rate is constant, I would like to approximate the real national income by constructing a discount factor satisfying $\tilde{\pi}(0) = 1$ and

$$\tilde{\pi}(s) = \exp\left(-\int_0^s R dv\right) = e^{-Rs}.$$

Still assume the real prices of consumption is constant as unity all the time. By replacing π with $\tilde{\pi}$ in (25), the real national income can be approximated by

$$\tilde{Y}(0) = E^0\left[\int_0^{\infty} \frac{\tilde{\pi}(s)}{\tilde{\pi}(0)} R(s) P_c(s) C(s) ds\right]$$

$$= E^0\left[\int_0^{\infty} e^{-Rs} R e^{\sigma B_s} k^{\alpha} ds\right]$$

$$= \frac{R}{R - \frac{1}{2}\sigma^2} k^{\alpha}. \quad (34)$$

Notice that by (32), the exact real national income in (33) can be rewritten as

$$Y(0) = \frac{R}{R - \frac{1}{2}\sigma^2 + \frac{1}{2}\eta\sigma^2} k^{\alpha}. \quad (35)$$

Then by (34) and (35), the approximation always overestimates the real national income. The rate of the error is given by

$$\frac{\tilde{Y}(0) - Y(0)}{Y(0)} = \frac{\frac{1}{2}\eta\sigma^2}{R - \frac{1}{2}\sigma^2},$$

which shows the approximation exactly coincides the real national income if there is no uncertainty, i.e. $\sigma = 0$, or the utility function is linear, i.e. $\eta = 0$. Otherwise, the approximation differs from the exact real national income.

By using the same values of parameters as in Figure **1**, we calculate the rates in Figure **2**[6]. Under certainty, i.e. $\sigma = 0$, the approximation coincides with the exact real national income. Under uncertainty, i.e. given certain level of $\sigma \neq 0$, higher curvature of utility function implies higher rates of the approximation error.

[6]In Figure **2**, the value of η is within the interval $[0,2]$ to highlight the tendency.

On the other hand, given the curvature of utility function η, the rates of the error is also increasing with higher level of uncertainty σ. Notice that the rate of the error is 40 percent of the exact real national income if $\eta = 2$ and $\sigma = 0.1$. It shows that the approximation is unacceptable under rather high curvature of utility function and uncertainty level. However, if the uncertainty level is small, then the approximation is quite good even though the curvature of utility function is rather high. For example, the rate of error is less than two percent of real national income if $\sigma = 0.01$ and $\eta = 10$.

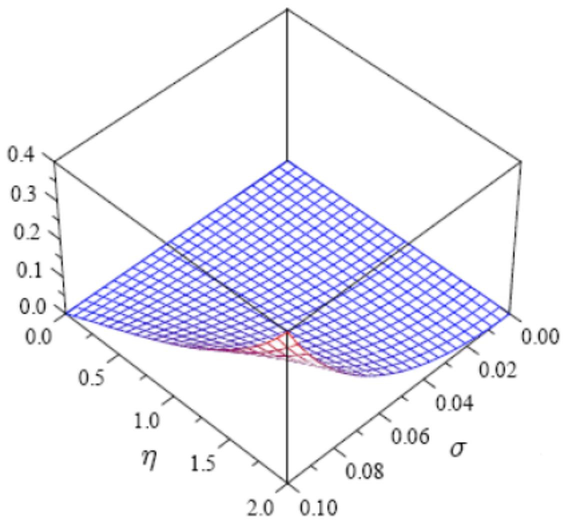

Figure 2: The rates of the error of approximation to the exact real national income w.r.t. the curvature of utility function and the uncertainty level of production.

After introducing a simple stochastic process into the production function, we find out that the theory under certainty is correct only under some special cases, e.g. the utility function takes some special forms. The linear utility function can not justify the extension of the theory from certainty to uncertainty. In our specific model, due to the expectation of higher future consumption, the real national income under uncertainty is higher than current production even though they are the same under certainty.

5. CONCLUDING REMARK

The paper studied the concept of national income (Sefton and Weale 2006, Asheim and Wei 2009) in the stochastic case. If the prices of consumption and net investment are properly defined, then stochastic national income defined here has plausible interpretation from both a welfare and productive respective. The concept of national income does not assume autonomous Itô diffusion for consumption even though it is required to show that real national income coincides with NNP.

Furthermore, stochastic *real* national income can be explained as the expectation of present value of real interests on future national consumption if a real consumption interest rate is properly defined.

A simple one-good model is used to illustrate the theory. The model shows that whether real national income equals current production or not depends on the future uncertainty and the curvature of utility function. Higher production does not necessarily imply higher real income if it is associated with more uncertainty. In the real world, there are great uncertainty that is not directly led by capital stocks. Then the deviation between real national income and NNP commonly exists.

Notice that the paper does not tell anything on how to find the proper prices and real consumption interest rate in the real world. Arrow, *et al.* (2003) has given some useful guidelines for practical calculation of these prices in the deterministic case. In addition, national income defined in the paper does not assume the optimal economic path.

APPENDIX

Proof of Proposition 1

$e^{-\rho(s-t)}$ is a deterministic process over time and $U(s)$ is an Itô process, then,

$$d\left[e^{-\rho(s-t)}U(s)\right] = e^{-\rho(s-t)}dU(s) - \rho e^{-\rho(s-t)}U(s)\,ds,$$

which is the representation of the integral

$$\int_t^\infty d\left[e^{-\rho(s-t)}U(s)\right] = \int_t^\infty e^{-\rho(s-t)}dU(s) - \int_t^\infty \rho e^{-\rho(s-t)}U(s)\,ds,$$

where the first term on the r.h.s. involves Itô integral. By rearranging terms, taking the expectation on both sides and applying the definition of welfare, we obtain that

$$E^t\left[\int_t^\infty e^{-\rho(s-t)}dU(s)\right] = -U(s) + \rho W(t), \qquad (36)$$

since $\lim_{s\to\infty} E\left[e^{-\rho(s-t)}U(s)\right] = 0$.

Let $h \geq 0$ be a very small time interval, then by the definition of welfare (10), the welfare at time $t+h$,

$$W(t+h) = E^{t+h}\left[\int_{t+h}^\infty e^{-\rho(s-(t+h))}U(s)\,ds\right]. \qquad (37)$$

On the other hand, the current welfare at time t can be rewritten as

$$W(t) = E^t\left[\int_t^{t+h} e^{-\rho(s-t)}U(s)\,ds + \int_{t+h}^\infty e^{-\rho(s-t)}U(s)\,ds\right]$$

$$= E^t\left[\int_t^{t+h} e^{-\rho(s-t)}U(s)\,ds\right] + E^t\left[\int_{t+h}^{\infty} e^{-\rho(s-t)}U(s)\,ds\right]$$

$$= E^t\left[\int_t^{t+h} e^{-\rho(s-t)}U(s)\,ds\right] + e^{-\rho h}E^t$$

$$\left[E^{t+h}\left[\int_{t+h}^{\infty} e^{-\rho(s-(t+h))}U(s)\,ds\right]\right]$$

$$= E^t\left[\int_t^{t+h} e^{-\rho(s-t)}U(s)\,ds\right] + e^{-\rho h}E^t\left[W(t+h)\right] \text{ by (37)} \quad (38)$$

for any $h \geq 0$. By the definition of the limit, we know that the limit of the first term on the r.h.s. of (38)

$$\lim_{h\to 0^+} \frac{E^t\left[\int_t^{t+h} e^{-\rho(s-t)}U(s)\,ds\right]}{h} = E^t\left[e^{-\rho(s-t)}U(s)\right]|_{s=t} = U(t). \quad (39)$$

Now directly by (38) and (39), the following expression holds

$$\lim_{h\to 0^+} \frac{E^t\left[W(t+h)\right] - W(t)}{h}$$

$$= -\lim_{h\to 0^+} \frac{E^t\left[\int_t^{t+h} e^{-\rho(s-t)}U(s)\,ds\right]}{h} + \lim_{h\to 0^+} \frac{\left(1-e^{-\rho h}\right)E^t\left[W(t+h)\right]}{h}$$

$$= -U(s) + W(t)\lim_{h\to 0^+} \frac{\left(1-e^{-\rho h}\right)}{h}$$

$$= -U(s) + \rho W(t). \quad (40)$$

Combine (11), (36) and (40), the proposition is proven. [Q.E.D.]

Proof of Corollary 1

Since the consumption $\{\mathbf{C}(s)\}_{s=0}^{\infty}$ is an Itô process, the property of chain rule of Stratonovich integral tells us

$$du(c(s)) = \nabla u(c(s)) \circ d\mathbf{C}(s) = e^{\rho(s-t)}\mathbf{p}_c(s) \circ d\mathbf{C}(s)$$

by (13). Since the utility $\{U(t)\}_{t=0}^{\infty}$ is also an Itô process under the settings, then Proposition 1 is valid. By (12), the *instantaneous welfare improvement,*

$$AW(t) = E^t\left[\int_t^{\infty} e^{-\rho(s-t)}\left(e^{\rho(s-t)}\mathbf{p}_c(s) \circ d\mathbf{C}(s)\right)\right]$$

$$= E^t\left[\int_t^{\infty} \mathbf{p}_c(s) \circ d\mathbf{C}(s)\right]$$

since $e^{\rho(s-t)}$ is a deterministic process over time. [Q.E.D.]

Proof of Proposition 2

By (18), the expected instantaneous change of welfare

$$AW(t) = \lim_{h\to 0^+} \frac{E^t\left[W(t+h)\right] - W(t)}{h} = \frac{E^t\left[dW(t)\right]}{dt}$$

$$= \frac{E^t\left[\nabla \tilde{w}(\mathbf{K}(t)) \circ d\mathbf{K}(t)\right]}{dt} = \frac{E^t\left[\mathbf{p}_k(t) \circ d\mathbf{K}(t)\right]}{dt}.$$

Since the consumption $\mathbf{C}$ and the utility U are also Itô processes by assumptions, then Corollary 1 is valid. By (14) and (15), the proposition is proved [Q.E.D.]

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